Problems related to the concentration of eigenfunctions

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Survey of joint work with Matthew Blair,
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Setting and general problem

Compact boundaryless manifold (M, g) of dimension $n \geq 2$.

Eigenfunctions:

$$-\Delta e_j(x) = \lambda_j^2 e_j(x), \qquad \int |e_j|^2 \, dV = 1$$

Give fundamental modes of vibration: $u_j(t,x) = \cos t\lambda_j \, e_j(x)$.

Vague Question: How can you detect and measure various types of concentration of eigenfunctions (or, more generally, quasi-modes)?

As $u_j(t,x)$ provide high-frequency solutions of wave equations, $(\partial_t^2 - \Delta)u_j = 0$, *expect answer to depend on long-term dynamics of geodesic flow* (e.g., propagation of singularities for $\partial_t^2 - \Delta$)

"Global harmonic analysis" or Harmonic/Globlal analysis

Extreme behavior on round spheres, S^n

Consider the standard sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : \, x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

Eigenvalues of $\sqrt{-\Delta_{S^n}}$ are

$$\sqrt{k(k+n-1)} \approx k,$$

repeating with highest possible multiplicity

$$d_k \approx k^{n-1}$$

(very non-generic).

Eigenfunctions are *spherical harmonics*, restrictions of homogeneous harmonic polynomials in $\mathbb{R}^{n+1}$ to S^n.

Extreme concentration at points

L^2-normalized zonal functions $Z_k(x)$, by classical Darboux-Szegö formula:

$$Z_k(x) \approx \cos\big((k + \tfrac{n-1}{2})d(x, \pm\mathbb{1}) + \sigma_n\big) / \big((d(x, \pm\mathbb{1})\big)^{\frac{n-1}{2}}, \text{ if } d(x, \pm\mathbb{1}) \geq k^{-1}$$

and $|Z_k(x)| = O(k^{\frac{n-1}{2}})$ if $d(x, \pm\mathbb{1}) \leq k^{-1}$, where

$$\mathbb{1} = (1, 0, \ldots, 0)$$

denotes north pole and $d(x, y)$ distance on S^n and $\sigma_n = -(n-1)\pi/4$ (Maslov factor).

High concentration at poles $\pm\mathbb{1}$.

Easy calculation using above:

$$\|Z_k\|_{L^p(S^n)} \approx k^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}, \quad p \geq \tfrac{2(n+1)}{n-1}. \tag{1}$$

Extreme concentration along periodic geodesics

Highest weight spherical harmonics,

$$Q_k(x) \approx k^{\frac{n-1}{4}}(x_1 + ix_2)^k$$

have extreme concentration near equator (periodic geodesic)

$$\gamma = \{x \in S^n : 0 = x' = (x_3, \dots, x_{n+1})\}.$$

Simplest example of "Gaussian beams",

$$|Q_k(x)| \approx k^{\frac{n-1}{4}} e^{-\frac{k}{2}d(x,\gamma)^2} \approx k^{\frac{n-1}{4}} \mathbb{1}_{\mathcal{T}_{k^{-\frac{1}{2}}}(\gamma)},$$

where $\mathcal{T}_{k^{-\frac{1}{2}}}(\gamma)$ denotes a $k^{-\frac{1}{2}}$ tubular neighborhood about γ.
Since equator has codimension $(n-1)$ conclude

$$\|Q_k\|_{L^p(S^n)} \approx k^{\frac{n-1}{4}} |\{x \in S^n : \, d(x,\gamma) \le k^{-\frac{1}{2}}\}|^{\frac{1}{p}} \approx k^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})}, \;\; p \ge 2 \quad (2)$$

Summary $L^p(S^n)$-norms of eigenfunctions

Note that for the critical value of p, $p_c = \frac{2(n+1)}{n-1}$, have

$$\|Z_k\|_{L^{\frac{2(n+1)}{n-1}}(S^n)} \approx \|Q_k\|_{L^{\frac{2(n+1)}{n-1}}(S^n)} \approx k^{\frac{n-1}{2(n+1)}}.$$

For larger exponents $p > p_c$, Z_k has larger L^p-norms, while for smaller ones $p < p_c$, Q_k wins.

Showed in my 1985 thesis on harmonic analysis on spheres that these are the worst case, i.e., if e_k spherical harmonic of degree k:

$$\|e_k\|_{L^p(S^n)} \lesssim k^{\sigma(p)} \|e_k\|_{L^2(S^n)} \tag{3}$$

$$\sigma(p) = \begin{cases} n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & p \geq \frac{2(n+1)}{n-1} \\ \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 < p \leq \frac{2(n+1)}{n-1} \end{cases} \tag{4}$$

Bounds for *"large"* p sensitive to *high concentration at points* and *"small"* p to *high concentration along periodic geodesics*

L^p-estimates for general compact manifolds

Motivated by potential applications in harmonic analysis and PDEs, showed that the above estimates (3)-(4) hold for all n-dimensional compact manifolds (M, g):

$$\|e_\lambda\|_{L^p(M)} \lesssim \lambda^{\sigma(p)} \|e_\lambda\|_{L^2(M)} \qquad (5)$$

Same bounds hold if f is a function whose $\sqrt{-\Delta_g}$-spectrum lies in unit interval $[\lambda, \lambda + 1]$.

Latter always sharp, but don't expect (5) to typically be saturated by eigenfunctions.

Use "global harmonic analysis" based on long-time dynamics of geodesic flow to try to see when (5) can be improved.

Generic improvements for large exponents ($p > p_c$)

Given $x \in M$ and an initial unit direction $\xi \in S_x^* M$ over x, say that $\xi \in \mathcal{L}_x$ if geodesic with initial direction ξ loops back through x in some positive time t.

(CS, S. Zelditch 2002) If $|\mathcal{L}_x| = 0$ for all $x \in M$ (a generic condition), then $\|e_\lambda\|_{L^\infty(M)} = o(\lambda^{\frac{n-1}{2}})$, and hence

$$\|e_\lambda\|_{L^p(M)} = o(\lambda^{\sigma(p)}), \quad \forall p > p_c = \tfrac{2(n+1)}{n-1} \tag{6}$$

Much stronger results if one considers *real analytic* manifolds: Let $\mathcal{C}_x \subset S_x^* M$ denote the subset of initial direction of smoothly closed (i.e., periodic) geodesics.

(CS, S. Zelditch 2014) If $n = 2$ have (6) for quasimodes if and only if there is no point $x \in M$ for which $\mathcal{C}_x = S_x^* M$. Also a nec/suff dynamical condition in higher dimensions.

Ideas in proofs of o-improvements for large exponents $(p > p_c)$

Can use propagation of singularities for wave operators to adapt proof of improvements in error term in Weyl law of Duistermaat-Guillemin/Ivrii (a trace estimate) to obtain improved L^∞ estimate (pointwise estimate).

Implicit in Bérard 1978: If (M, g) has nonpositive sectional curvatures $\|e_\lambda\|_{L^\infty(M)} = O(\lambda^{\frac{n-1}{2}}/\sqrt{\log \lambda})$ (i.e.,log-improvement).

Hassell-Tacy 2011: Also get the same log-improvements over (6) for all exponents $p > p_c$ under this curvature assumption.

Problem: When can you get even just o-improvements of the eigenfunction estimates (3)-(4) for the critical exponent, $p_c = \frac{2(n+1)}{n-1}$?

Remember this exponent sees both concentration at points **AND** concentration along periodic geodesics.

Other ways of measuring concentration along geodesics

We've argued that the L^p-estimates (5) for $2 < p < p_c$ are sensitive to concentration along geodesics as exhibited by the higher weight spherical harmonics, Q_k. What about other estimates?

- Lower bounds for L^1-norms (due to CS-Zelditch 2011):

$$\lambda^{-\frac{n-1}{4}} \lesssim \|e_\lambda\|_{L^1(M)}. \tag{7}$$

- Extreme L^2-concentration about $\lambda^{-\frac{1}{2}}$ tubes about unit length geodesics $\gamma \in \Pi$:

$$\|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma))} \le 1 \tag{8}$$

Both are saturated by the Q_k, $\lambda = \sqrt{k(k+n-1)}$ since Q_k behaves like $\lambda^{\frac{n-1}{4}}$ times indicator function of $\lambda^{-\frac{1}{2}}$ neighborhood of equator.

More about lower bounds for L^1-norms

The lower bound (7) actually follows from Hölder's inequality and the old L^p upper bounds (5) for any $2 < p \le p_c$:

$$1 = \|e_\lambda\|_2 \le \|e_\lambda\|_1^\theta \|e_\lambda\|_p^{1-\theta}, \text{ relevant } \theta = \theta(p)$$

Using this and the bound $\|e_\lambda\|_p = O(\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})})$ yields (7), i.e.,

$$\lambda^{-\frac{n-1}{4}} \lesssim \|e_\lambda\|_1.$$

Argument show that given (M,g) improved L^p-norms for $2 < p < p_c$ yield improved L^1 lower bounds.

Can show that if lower bound (7) is saturated then there must be a $\lambda^{-\frac{1}{2}}$ geod-tube and $\delta > 0$ and $0 < c < \infty$ (depending only on (M,g)) so that, as λ ranges of a subseq of e.v.'s, (just like for the Q_k) have

$$|\{x \in \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma) : |e_\lambda(x)| \in [c\lambda^{\frac{n-1}{4}}, c^{-1}\lambda^{\frac{n-1}{4}}]\}| \ge \delta |\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)|.$$

L^1-lower bounds and lower bounds for the size of nodal sets

Consider the nodal set of a given real eigenfunction,

$$Z_\lambda = \{x \in M : \, e_\lambda(x) = 0\}.$$

Known that Z_λ a smooth hypersurface off set of $(n-2)$ Hausdorff dimension. Let $|Z_\lambda| = \mathcal{H}^{n-1}(Z_\lambda)$ denote the $(n-1)$-dimensional Hausdorff measure of Z_λ.

Conjecture of Yau 1970s: $|Z_\lambda| \approx \lambda$

Completely solved in real analytic case by Donnelly-Fefferman c. 1990. Also, lower bound in C^∞ case when $n = 2$ obtained by Brüning-Yau 1970s. Upper bound $|Z_\lambda| = O(\lambda^{\frac{3}{2}})$ by Dong and Donnelly-Fefferman for smooth case when $n = 2$.
Best known upper bounds for $n \geq 3$ are doubly exponential (Hardt-Simon).

Until recently, best known lower bounds were due to Han-Lin $e^{-c\lambda} \lesssim |Z_\lambda|$.

Lower bounds for size of nodal sets, cont.

The exponential barrier for lower bounds for $|Z_\lambda|$ was broken independently by Colding-Minicozzi and Sogge-Zelditch in 2011.

Current world record for general case when $n \geq 3$ is due to C-M:

$$\lambda^{1-\frac{n-1}{2}} \lesssim |Z_\lambda|. \tag{9}$$

Their approach was to use "good ball" idea of Donnelly-Fefferman and L^{p_c}-bounds (3) to establish (9).

In other words, C-M were able to prove (9) by showing that eigenfunctions could not be "extremely concentrated" on sets of an appropriate scale.

Alternate approach of CS-Zelditch

CS-Zelditch obtained a Dong-type-identity

$$\lambda^2 \int_M |e_\lambda| \, dV = 2 \int_{Z_\lambda} |\nabla e_\lambda| \, dS.$$

In CS-Z 2011 we also proved the L^1-lower bounds (7) using the bound $\|e_\lambda\|_\infty \lesssim \lambda^{\frac{n-1}{2}} \|e_\lambda\|_1$. Same argument shows

$$\|\nabla e_\lambda\|_\infty \lesssim \lambda^{1+\frac{n-1}{2}} \|e_\lambda\|_1,$$

whence, by above,

$$\lambda^2 \|e_\lambda\|_1 \leq 2|Z_\lambda| \, \|\nabla e_\lambda\|_\infty \lesssim \lambda^{1+\frac{n-1}{2}} \|e_\lambda\|_1 \, |Z_\lambda|,$$

recovering the C-M world record

$$\lambda^{1-\frac{n-1}{2}} \lesssim |Z_\lambda|.$$

Variations on this theme and L^1 lower bounds, redux

Using the above Dong-identity and its proof, Hezari and CS showed that

$$\lambda\Big(\int_M |e_\lambda|\, dV\Big)^2 \lesssim |Z_\lambda|, \tag{10}$$

which also leads to the Colding-Minicozzi world record (9) if you use the L^1-lower bound of CS-Zelditch:

$$\lambda^{-\frac{n-1}{4}} \lesssim \|e_\lambda\|_{L^1(M)}.$$

Of course (10) shows that any improvements of this L^1-lower bound lead to improvements of the C-M world record.

Blair-CS showed that there are improved $L^p(M)$-norms for $2 < p < p_c$ and, hence, improved L^1 lower bounds (using Hölder as before) if (M, g) has nonpositive sectional curvatures. Hence in this case, B-CS showed that

$$\liminf_{\lambda\to\infty} \lambda^{-1+\frac{n-1}{2}} |Z_\lambda| = \infty.$$

L^2-mass on shrinking geodesic tubes, redux

Recall that we posited that another way of measuring an extreme concentration was in terms of L^2-mass over shrinking tubes, i.e., the "Kakeya-Nikodym" quanitites

$$\|e_\lambda\|_{KN} = \sup_{\gamma\in\Pi} \|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma))}.$$

Since our e.f.'s are L^2-normalized, one has the trivial bounds $\|e_\lambda\|_{KN} \le 1$. When $n = 2$ there is the related (but non-trivial) restriction estimate of Burq-Gérard-Tzvetkov saying that

$$\int_\gamma |e_\lambda|^2 ds \lesssim \lambda^{\frac{1}{2}}. \tag{11}$$

Any improvements over B-G-T lead to nontrivial KN bounds when $n = 2$. This was done by CS-Zelditch under the assumption of nonpos curv.

Restriction estimates turn out to be too singular in higher dimensions to yield improved KN-bounds. Still, under the assumption of nonpos curvature, Blair-CS beat the trivial KN bounds.

Kakeya-Nikodym estimates

Using 1970s classical analysis techniques of Hörmander and Cordoba-Fefferman showed that, when $n = 2$, in 2011 CS showed

$$\|e_\lambda\|_{L^4(M)} \lesssim \lambda^{\frac{1}{8}} \|e_\lambda\|_{L^2(M)}^{\frac{3}{4}} \times \sup_{\gamma \in \Pi} \|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma))}^{\frac{1}{4}}, \tag{12}$$

which improves the 1980s bounds $\|e_\lambda\|_4 \lesssim \lambda^{\frac{1}{8}}$.

A variation of this with B-G-T restriction norms in RHS (and an λ^ε loss) was done earlier in 2009 by Bourgain.

Immediately see that the aforementioned improved KN estimates for $n = 2$ lead to improved L^4-bounds (and hence improved L^p, $2 < p < 6$ by interpolation).

Using (12) and another estimate from Bourgain 2009 can show that three problems: 1) improved $L^4(M)$ estimates, 2) improved B-G-T geodesic restriction estimates and 3) improved KN estimates are equivalent.

Key identity: Designer eigenfunction reproducing formula

Choose $\rho \in \mathcal{S}(\mathbb{R})$ satisfying

$$\rho(0) = 1, \text{ and } \hat{\rho}(t) = 0, \ |t| \notin (\delta/2, \delta).$$

Then $\rho(\lambda - \sqrt{-\Delta_g})e_\lambda = e_\lambda$, and

$$\rho\big(\lambda - \sqrt{-\Delta_g}\big)(x,y) = \lambda^{\frac{n-1}{2}} e^{i\lambda d_g(x,y)} a_\lambda(x,y) \ + O(\lambda^{-N}),$$

where

$$|D^\alpha_{x,y} a_\lambda| \le C_\alpha, \quad \text{and } \ a_\lambda(x,y) = 0 \text{ if } d_g(x,y) \notin [\delta/2, \delta].$$

To prove improved results for nonpos curvature use $\rho(T(\lambda - \sqrt{-\Delta_g}))$, Hadamard parametrix for universal cover and

$$\rho(T(\lambda - \sqrt{-\Delta_g})) = \frac{1}{2\pi T} \int_{-T}^{T} \hat{\rho}(t/T)\, e^{i\lambda t}\, e^{-it\sqrt{-\Delta_g}}\, dt.$$

Higher-dimensional Kakeya-Nikodym estimates

Using more recent harmonic analysis techniques developed by Bourgain, Lee, Tao-Vargas, Blair-CS were able to expend (12) to higher dimensions

$$\|e_\lambda\|_{L^p(M)} \lesssim \lambda^{\sigma(p)} \|e_\lambda\|_{L^2(M)}^{1-\theta} \times \sup_{\gamma\in\Pi} \|e_\lambda\|^{\theta}_{L^2(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma))},$$
$$2 < p < p_c, \ \text{some} \ \theta = \theta_p \in (0,1),$$

which along with the aforementioned improved KN bounds for nonpos curv leads to improved L^p bounds, and hence improved L^1-lower bounds and improved lower bounds for size of nodal sets, as described before.

Refined Kakeya-Nikodym estimates in 2-dimensions

Recently, Blair-CS proved the somewhat better KN-estimates

$$\|e_\lambda\|_{L^4(M)} \lesssim_\varepsilon \lambda^{\frac{1}{8}} \|e_\lambda\|_{L^2(M)}^{\frac{1}{2}} \times \sup_{\gamma \in \Pi} \|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-\frac{1}{2}+\varepsilon}}(\gamma))}^{\frac{1}{2}}, \tag{13}$$

Better than its cousin (12) since, instead of powers $(\frac{3}{4}, \frac{1}{4})$, have powers $(\frac{1}{2}, \frac{1}{2})$. There is an apparent ε-loss, though.
Not so concerned with this "loss" since, if one could prove natural QE estimates (assuming negative curv...)

$$\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}+\varepsilon}}(\gamma))} |e_\lambda|^2 dV \lesssim |\mathcal{T}_{\lambda^{-\frac{1}{2}+\varepsilon}}(\gamma)| \approx \lambda^{-\frac{1}{2}+\varepsilon}, \tag{14}$$

would get natural analog of Zygmund's $L^4(\mathbb{T}^2)$ bounds, saying, that, here,

$$\|e_\lambda\|_{L^4(M)} \lesssim_\varepsilon \lambda^\varepsilon, \quad \forall \varepsilon > 0.$$

Of course above shrinking scale QE estimates seem very difficult.

Recent small-scale quantum ergodic estimates

Although estimates like (14) involving QE estimates on λ-power scales seem very difficult, in 2014, X. Han and H. Hezari and G. Rivière were able to prove related bounds on logarithmic scales:

Theorem (X. Han and H. Hezari and G. Rivière)

Let (M, g) be negatively curved. Then if $\{e_{\lambda_j}\}$ is an orthonormal basis of eigenfunctions, there is a density one subsequence of eigenvalues, $\{\lambda_{j_k}\}$, and $\alpha_n, c > 0$ so that

$$cr^n \le \int_{B_r(x)} |e_{\lambda_{j_k}}|^2 \, dV \le c^{-1} r^n, \quad \forall x \in M, \text{ if } r = (\log \lambda)^{-\alpha_n}. \tag{15}$$

H. Hezari and G. Riviére used (15) and localized eigenfunction bounds to show that this subsequence satisfies the improved critical L^{p_c}-bounds

$$\|e_{\lambda_{j_k}}\|_{L^{p_c}} = O(\lambda_{j_k}^{\frac{1}{p_c}} / (\log \lambda_{j_k})^{1/\sigma_n}), \ p_c = \tfrac{2(n+1)}{n-1}.$$

Improved ball-localized eigenfunction estimates

Can improved the ball-localized estimates of Hezari and Riviére. Just by using original 1988 spectral projection bounds, get:

Theorem

If $\|e_\lambda\|_2 = 1$, then there is a constant $C = C(M,g)$ so that for all $r > 0$ smaller than the injectivity radius,

$$\|e_\lambda\|_{L^{\frac{2(n+1)}{n-1}}(M)} \le C\lambda^{\frac{n-1}{2(n+1)}} \left(r^{-\frac{n+1}{4}} \sup_{x\in M} \|e_\lambda\|_{L^2(B_r(x))}\right)^{\frac{2}{n+1}}. \qquad (16)$$

Corollary

If the geodesic flow on (M,g) is ergodic, then there is a density one sequence of eigenvalues, $\{\lambda_{j_k}\}$, so that

$$\|e_{\lambda_{j_k}}\|_{L^{\frac{2(n+1)}{n-1}}(M)} = o\big(\lambda_{j_k}^{\frac{n-1}{2(n+1)}}\big).$$

Open problems: nontrivial $L^2(B_r)$ estimates

Easy to see that for $0 < r \ll 1$, have uniform bounds

$$\|e_\lambda\|_{L^2(B_r(x))} \le Cr^{\frac{1}{2}}. \qquad (17)$$

This estimate is saturated by zonal functions for all r (and by highest weight spherical harmonics if $0 < r \ll \lambda^{-\frac{1}{2}}$).

Question: When can you beat the trivial bound (17)?

Seems difficult. Techniques used to prove non-trivial L^2 bounds over small tubes for nonpositive curvature don't seem to apply.

Using Bérard's log-improved sup-norm estimates for nonpos curvature, *can show that get log-improvements for* (16) *when* $r = \lambda^{-1}$, and then use this fact and L^∞ variation of localized L^p-bounds to *recover his sup-norm estimates.* But latter argument is circular.

Can also recover $\|Q_k\|_{L^p(S^n)}$ by using (16) and knowledge of $\|Q_k\|_{L^2(B_r(x))}$.