Chris Sogge - Concentration of Eigenfunctions 08/07/2015.

(M,g) cpt Riem, dim ≥ 2.

$$-\Delta e_j(x) = \lambda_j^2 e_j(x) \qquad \int |e_j|^2 dV = 1.$$

Modes of vibration, $u_j(t,x) = \cos t\lambda_j \, e_j(x)$.

Q: Detect and measure types of concentration of eigenfunctions

Harmonic analysis to ~~do~~ locally + Wave techniques to do global.

Extreme behavior on S^n.

Eigenvalues of $\sqrt{-\Delta_{S^n}}$ are $\sqrt{k(k+n-1)} \approx k$.

multiplicity $d_k = k^{n-1}$, highest possible b/c of Weyl's law.

Spherical harmonics are eigenfunctions.

Highest weight spherical harmonics $Q_k(x) \approx k^{\frac{n-1}{4}}(x_1 + i x_2)^k$.

have extreme concentration near equator. ~~Eg~~.

Gaussian beam: $|Q_k(x)| \approx k^{\frac{n-1}{4}} e^{-\frac{k}{2} d(x,\gamma)^2}$. γ - equator.

Estimates of the form $\|e_\lambda\|_{L^p(M)} \lesssim \lambda^{\sigma(p)} \|e_\lambda\|_{L^2(M)}$.

where $\sigma(p)$ splits into two cases with critical scale.
using Bochner-Riesz summability, $p_c = \frac{2(n+1)}{n-1}$.

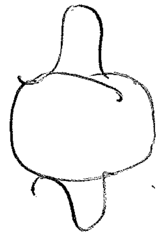
Improvements for exponents $p > p_c$.

$x \in M$, $\xi \in S^*_x M$, Say $\xi \in \mathcal{L}_x$ if $\exists$ geodesic γ_ξ s.t. $\exists\, t'$ so that $\gamma_\xi(t') = \gamma_\xi(0) = x$. (ie loops back).

If $|\mathcal{L}_x| = 0$ $\forall x \in M$, then $\|e_\lambda\|_{L^\infty(M)} = o(\lambda^{\frac{n-1}{2}})$, so
$\|e_\lambda\|_{L^p(M)} = o(\lambda^{\sigma(p)})$, $p > p_c$.

<u>Note:</u> on $M = S^n$, $\mathcal{L}_x = S^*_x M$ because ~~of~~ every geodesic is a great ~~sph~~ circle and these have conjugate points and loop back in 2π-mod.

Large p: pick up concentration on pts:



Small p: pick up concentrations in periodic geo.

<u>Eigenfunction eigenfunctions:</u> Impossible to do directly in gen.

$\rho \in \mathcal{S}(\mathbb{R})$, s.t. $\rho(0) = 1$, $\hat{\rho}(t) = 0$, $|t| \notin (\delta/2, \delta)$.

Spectral Th$^{\underline{m}}$: $\rho(\lambda - \sqrt{-\Delta_g})e_\lambda = e_\lambda$,

$$\rho(\lambda - \sqrt{-\Delta_g})(x,y) = \lambda^{\frac{n-1}{2}} e^{i\lambda d_g(x,y)} a_\lambda(x,y) + O(\lambda^{-N}).$$

$|D^\alpha_{x,y}| \le C_\alpha$, and $a_\lambda(x,y) = 0$ if $d_g(x,y) \notin [\delta/2, \delta]$

Improved results for non-pos curv, $\rho(T(\lambda - \sqrt{-\Delta_g}))$,

<u>Hadamard parametrix for universal cover</u> (?) etc.

$$\rho(T(\lambda - \sqrt{-\Delta_g})) = \frac{1}{2\pi T}\int_{-T}^{T} \hat{\rho}(t/T)\, e^{i\lambda t} e^{-it\sqrt{-\Delta_g}}\, dt$$

$\times$ Hille - Phillips ~~...~~ f.c. right?

Slides available on website. (personal).