Kakeya-Nikodym estimates for eigenfunctions

Chris Sogge (Johns Hopkins University)

Joint work with:

Matthew Blair (University New Mexico)
& Steve Zelditch (Northwestern University)

Xuehua Chen (Northwestern University)

Overview

Let (M, g) be a 2-dimensional compact Riemannian manifold with eigenfunctions

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda, \quad \int_M |e_\lambda|^2 \, dV_g = 1.$$

Basic questions: When do high frequency eigenfunctions concentrate on lower dimensional sets? When not? Role of geometry/dynamics? How can you measure this?

1. Kakeya-Nikodym estimates, $L^4(M)$-norms, and quantum scarring. L^4-norms as *scar detectors.*
2. Small L^4-norms and no extreme scarring for nonpositive curvature. Period integrals.

L^p norms of eigenfunctions

Let $\sigma(p) = \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$ if $2 < p \le 6$ and $\sigma(p) = 2(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$ if $2 \le p \le \infty$. Then

$$\|e_\lambda\|_p \lesssim \lambda^{\sigma(p)} \|e_\lambda\|_2.$$

- Sharp for **sphere**
- Also sharp for **any** manifold if, instead of eigenfunctions, you measure bounds for functions whose spectrum lie in unit bands, $[\lambda, \lambda + 1]$. So one of our goals is to see when eigenfunctions do not look like these "quasimodes".
- CS was interested in above estimate for $p \ge 6$ since then $\sigma(p)$ is the critical index for Bochner-Riesz. Used these bounds to prove Böchner-Riesz bounds, as well as Hörmander multiplier theorem (latter with Seeger, $p = \infty$ relevant).
- Until recently, bounds for other range $2 < p < 6$ were "afterthought". Turn out to be *key for above questions*.
- Special case, $\|e_\lambda\|_4 \lesssim \lambda^{\frac{1}{8}}$.

Proof of L^p norms of eigenfunctions

Use *reproducing formula*, $e_{\lambda_j} = \rho(\lambda_j - \sqrt{-\Delta_g})e_\lambda$, if $\rho(0) = 1$

$$\rho(\lambda_j - \sqrt{-\Delta_g})f = \sum_k \rho(\lambda_j - \lambda_k)E_k f, \ E_k = \text{proj on eigensp w/ e.v. } \lambda_k$$

For if $f = e_{\lambda_j}$ all terms 0 except $k = j$ term which is $\rho(0)e_{\lambda_j} = e_{\lambda_j}$.
If $\rho \in \mathcal{S}$ and $\hat{\rho}(t) = 0$, $t \not\approx \delta = \mathrm{Inj}(M)/2$ use formula

$$\rho(\lambda - \sqrt{-\Delta_g}) = \frac{1}{2\pi}\int_{-\delta}^{\delta} \hat{\rho}(t) e^{i\lambda t} e^{-it\sqrt{-\Delta_g}}\, dt$$

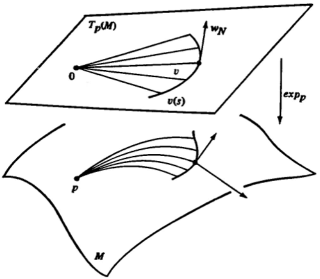
and small-time wave operator info to see that kernel is
$\approx \lambda^{\frac{1}{2}} a(d_g(x,y))\, e^{i\lambda d_g(x,y)}$ w/ d_g Riemannian distance, $a \in C^\infty(\mathbb{R})$.

Finish proof with Hörmander/Stein's oscillatory integral theorem (and Gauss' lemma).

This **local approach** cannot yield L^p-improvements. Instead need to use "dilated" reproducing operators $\rho(T(\lambda - \sqrt{-\Delta_g}))$, $T \gg 1$.
Hard since need to understand wave and geodesic flow dynamics up to times $\approx T$. Need: Global harmonic analysis.

Relevance of Gauss lemma (geometry)

Gauss: Given $p \in M$, $\exp_p : T_pM \to M$ diffeo near 0 & images of spheres of small radius in $T_pM \perp$ to all geodesics starting at p.



Google translation: This and the fact that $y \to \nabla_x d_g(x,y) = \{\xi : \sum g^{jk}(x)\xi_j\xi_k = 1\}$ is convex allows you to control *certain 2nd and 3rd mixed derivatives of distance function*: Rank $\nabla_x \nabla_y d_g(x,y) \equiv n-1$, and if γ geodesic ray starting at x and $v = \dot\gamma(0)$, $\nabla_y \langle v, \nabla_x\rangle d_g(x,y) = 0$, $y = \gamma(t)$, but, if $\mu \nparallel \dot\gamma(t)$, $(\langle \mu, \nabla_y\rangle)^2 \langle v, \nabla_x\rangle d_g(x,y) \neq 0$, $y = \gamma(t)$.

I) Kakeya-Nikodym estimates

Link "global" problem of improved L^4-bounds with familiar "local object": Let Π denote the space of unit-length geodesics in M, and $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ a $\lambda^{-\frac{1}{2}}$ tube about $\gamma \in \Pi$ (points dist $\lambda^{-\frac{1}{2}}$ from γ. Then:

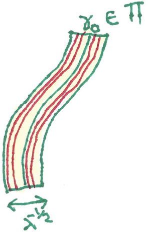
Theorem 1.(Bourgain 2009, CS 2011). Given (M, g) T.F.A.E.:

1. $\|e_\lambda\|_4 = o(\lambda^{\frac{1}{8}})$
2. $\sup_{\gamma \in \Pi} \int_\gamma |e_\lambda|^2 ds = o(\lambda^{\frac{1}{2}})$
3. $\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |e_\lambda|^2 \, dV_g = o(1)$

Burq, Gérard, Tzvetkov (2007) (cf. Reznikov): One has **"restriction estimates"**: $\int_\gamma |e_\lambda|^2 \, ds = O(\lambda^{\frac{1}{2}})$ (& sharp for S^2).

Obviously 2) $\implies$ 3). Bourgain: 1) $\implies$ 2). CS 3) $\implies$ 1)
CS also showed that automatically have $\int_\gamma |e_\lambda|^2 \, ds = o(\lambda^{\frac{1}{2}})$ if γ not unit segment of periodic geodesic.

Tube estimates vs Restriction estimates



Can foliate $\lambda^{-1/2}$-tube $\mathcal{T}_{\lambda^{-1/2}}(\gamma_0)$

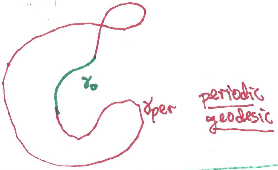
by $O(\lambda^{-1/2})$-width of geodesics $\gamma \approx \gamma_0$

Thus,

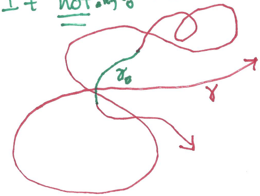
$$\int_\gamma |e_\lambda|^2 ds \le \varepsilon \lambda^{1/2} \Rightarrow \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma_0)} |e_\lambda|^2 dV_g \lesssim \varepsilon$$

Only possible bad case

$\gamma_0 \in \gamma_{per} = \gamma$



If not, i.e. $\gamma_0 \in \gamma$ not periodic



$$\Rightarrow \int_\gamma |e_\lambda|^2 ds = o(\lambda^{1/2})$$

$$\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma_0)} |e_\lambda|^2 dV_g = o(1)$$

A Kakeya-Nikodym inequality

$$\|e_\lambda\|_{L^4(M)} \le C\lambda^{\frac{1}{8}} \|e_\lambda\|_{L^2(M)}^{\frac{3}{4}} \sup_{\gamma\in\Pi} \|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma))}^{\frac{1}{4}}. \qquad (1)$$

- Bourgain: earlier version with $\lambda^{\frac{1}{8}+\varepsilon}$, $\forall\varepsilon$.
- Above has right power of λ, but not happy with red powers. Córdoba's BR work and Mockenhaupt-Seeger-CS suggests $(\frac{1}{2}, \frac{1}{2})$.
- Issues with getting this, since in Córdoba's approach (or MSS) maximal operators go on square functions coming from angular decomposition, and not just on e.g., $(e_\lambda)^2$.

Above KN estimate follows from arithmetic and CS estimate:

$$\int |e_\lambda|^4\, dV_g \le C_0 N^{-1}\lambda^{\frac{1}{4}} \|e_\lambda\|_{L^2(M)}^2 \|e_\lambda\|_{L^4(M)}^2 + C_0 N\lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(M)}^2 \Big(\sup_\gamma \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |e_\lambda|^2\, dV_g\Big),\ \forall N \qquad (2)$$

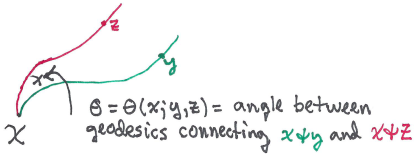
Proof: Merging Hörmander's and Córdoba's approach

$\chi_\lambda e_\lambda = \rho(\lambda - \sqrt{-\Delta_g})e_\lambda = e_\lambda$, $\chi_\lambda(x,y) = \lambda^{\frac{1}{2}} e^{i\lambda d_g(x,y)} a(x,y)$, where $a(x,y) = 0$ unless $d_g(x,y) \approx \delta$. Take $f = e_\lambda$. Want:

$$\int (\chi_\lambda f)^2 \, \overline{f^2} \lesssim N^{-\frac{1}{2}} \lambda^{\frac{1}{4}} \|f\|_2^2 \|f\|_4^2 + N\lambda^{\frac{1}{2}} \|f\|_2^2 \times \sup \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |f|^2.$$

$$(\chi_\lambda f)^2 = \lambda \int_{\theta \geq N\lambda^{-\frac{1}{2}}} + \int_{\theta \leq N\lambda^{-\frac{1}{2}}} e^{i\lambda(d_g(x,y)+d_g(x,z))} a(x,y)a(x,z)f(y)f(z)\,dydz$$

$$= H_N(f \otimes f)(x) + C_N(f \otimes f)(x).$$



⋆Hörmander: $\|H_N(f \otimes f)\|_2 \lesssim \lambda \lambda^{-\frac{3}{4}} N^{-\frac{1}{2}} \|f\|_2^2$. (1st term in RHS)

⋆If you replace $(\chi_\lambda f)^2$ by $C_1(f \otimes f)$ (i.e., y and z in same $\lambda^{-1/2}$-sectors about x), dominated by 2nd term in RHS w/ $N = 1$. General case $N \in \mathbb{N}$ by Cauchy-Schwarz and Gauss for this.

With Blair/Zelditch: Refined and microlocal KN bounds

Theorem 2. Given $0 < \varepsilon_0 \leq \frac{1}{2}$,

$$\|e_\lambda\|_{L^4(M)} \lesssim_{\varepsilon_0} \lambda^{\frac{\varepsilon_0}{4}} \|e_\lambda\|_{L^2(M)}^{\frac{1}{2}} \times \Big[\sup_{\gamma \in \Pi} \big(\lambda^{\frac{1}{2}-\varepsilon_0} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}+\varepsilon_0}}(\gamma)} |e_\lambda|^2 \, dV_g \big)^{\frac{1}{2}} \Big]^{\frac{1}{2}} \tag{3}$$

Remarks: If $\varepsilon_0 = \frac{1}{2}$, this is just 1980s CS theorem, $\|e_\lambda\|_4 \lesssim \lambda^{\frac{1}{8}}$. If powers $(\frac{1}{2}, \frac{1}{2})$ replaced by worse ones $(\frac{3}{4}, \frac{1}{4})$, but $\varepsilon_0 = 0$ (and no loss), this is above CS 2011 KN estimate (1) above. Not sure, whether we can push above down to $\varepsilon_0 = 0$. Would be sharp. We prove (3) using microlocal analysis and obtaining a *stronger estimate* where supremum in right is replaced by

$$\theta_0^{-\frac{1}{2}} \sup_{\gamma \in \Pi} \|Q_\gamma^{\theta_0}(x, D) e_\lambda\|_{L^2(M)}, \quad \theta_0 = \lambda^{\frac{1}{2}-\varepsilon_0},$$

and the PDOs Q_γ^θ denote natural ones with symbols living on θ tubes about the geodesic in S^*M.

Can use this stronger microlocal KN result to recover result of CS-Zelditch that generic e.f.'s have $L^4(M)$ norms of size $o(\lambda^{\frac{1}{8}})$

Model case: Relations with Zygumund's $L^4(\mathbb{T}^2)$-thm

A motivation for obtaining inequalities like (3) w/ improved powers on sup is potential applications to bnds arising in number theory. S. Marshall, P. Sarnak ... have begun using restriction estimates....

Consider the 2-torus, and suppose that one has uniform bounds

$$\int_\gamma |e_\lambda|^2 \, ds \le C, \qquad (4)$$

for every unit geodesic γ in $\mathbb{T}^2 \simeq [-\pi, \pi)^2$ and L^2-normalized toral e.f. $e_\lambda = \sum_{\{k \in \mathbb{Z}^2 : |k| = \lambda\}} a_k e^{ik \cdot x}$.

By earlier observation, (4), implies supremum in (3) is $O(1)$ and so conclude $\|e_\lambda\|_{L^4(\mathbb{T}^2)} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0}$ for any $\varepsilon_0 > 0$.

If (3) were valid with $\varepsilon_0 = 0$ and (4) were valid, we'd recover **Zygmund's theorem:** $\|e_\lambda\|_4 \le C$.

Recent observation of Sarnak: Have (4) **iff** the number of lattice points in $\mathbb{Z}^2$ on arcs of length $\lambda^{\frac{1}{2}}$ of λS^1 is O(1). (Result of Cilleruelo-Córdoba (1992) says OK for arcs length $\lambda^{\frac{1}{2} - \delta}$.)

Quantum unique ergodicity versus scarring

L^2-normalized e.f.'s define a probability measures $|e_\lambda|^2\, dV_g$. Say that we have quantum unique ergodicity for (M, g) if the weak$*$ limit of these measures is the uniform measure $dV_g/|M|$.

Special case: $\Omega \subset M$ (good) open set have

$$\int_\Omega |e_\lambda|^2\, dV_g \to \frac{|\Omega|}{|M|}. \tag{5}$$

If we don't have $|e_\lambda|^2\, dV_g \rightharpoonup dV_g/|M|$, say there is **scarring**.

More natural, to consider "microlocal measures"

$$a \in C^\infty(S^*M) \to \langle a(x, D)e_\lambda, e_\lambda\rangle = \int_{S^*M} a(x, \xi)\, d\mu_\lambda,$$

which are the "microlocal lifts" of $|e_\lambda|^2 dV_g$. QUE if these measures tend weekly to uniform Liouville prob msr on S^*M.

Conjecture Rudnick & Sarnak: QUE if curvatures negative. (Only known in very special cases (arithmetic), e.g. Lindenstrauss.)

Quantum unique ergodicity vs scarring, cont.

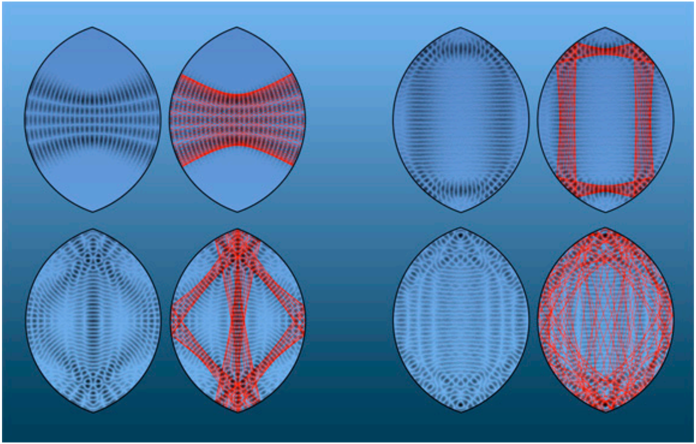
Known by Shnirelman / Colin de Verdiére / Zelditch that if geodesic flow in S^*M is ergodic (automatic for neg curv) then have that microlocal lifts $d\mu_{\lambda_{j_k}}$ tend to Liouville prob measure for a subsequence of e.v.'s, $\{\lambda_{j_k}\}$, of density one. So, in this case, if (5) breaks down, must do so in very sparse subsequence of e.v.'s.

Also known that if a measure on S^*M is in the limit set, must be invariant under geodesic flow.

Natural question: Can you rule out for *neg curv* $|e_{\lambda_{j_k}}|^2 dV_g$ tending through subsequence of e.v.'s to linear combination of delta-measure, $ds_{\gamma_{per}}$ on periodic geodesic and another invariant measure?

Anantharaman (2008): Limit for neg curvature cannot just be probability measure on γ_{per} (i.e., $ds_{\gamma_{per}}/\text{Length}\ (\gamma_{per})$), or a finite combination of such.

At this stage can't rule out certain combinations of these and uniform measure on S^*M, though.



Left pictures: Unstable periodic orbits.

Right pictures: Wave functions $|e_\lambda|^2$ are large superimposed over these orbits.

Source: Eric J. Heller, Harvard Physics Dept. (www.ericjhellergallery.com)

Extreme scarring

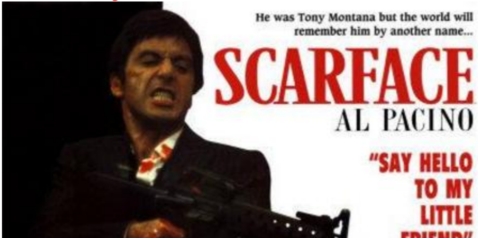
The normalized "highest weight" sph. har. $Q_k \approx k^{\frac{1}{4}}(x_1 + ix_2)^k$ on $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ Satisfy

$$|Q_k| \approx k^{\frac{1}{4}} \exp(k \ln((1 - x_3^2)^{1/2})) \approx k^{\frac{1}{4}} e^{-kx_3^2/2},$$

(Gaussian beams), whence, $|Q_k|^2 dV \approx k^{\frac{1}{2}} e^{-kx_3^2} dV$ tends to delta measure on equator, $x_3 = 0$. So cannot have (5) (for instance) if $\Omega \subset S^2$ is disjoint from equator.

Note that Q_k e.f.'s w/ e.v. $\sqrt{k(k+n-1)} \approx k$, and $\|Q_k\|_{L^4(S^2)} \approx k^{\frac{1}{8}}$, and they have $\Omega(1)$ $L^2(dV)$-mass in a $k^{-\frac{1}{2}}$-tube about equator (a periodic geodesic on S^2).

Worst possible enemy:



$L^4(M)$-norms as **scarring detectors**

$\|e_\lambda\|_{L^4(M)} = \Omega(\lambda^{1/8})$ implies **must have scarring:**

Specifically, can find a geodesic segment, γ_0, of length 2 so that if $\Omega = \mathcal{T}_\delta(\gamma_0)$ $\exists$ subsequence of e.v.'s $\{\lambda_{j_k}\}$ so that for small $\delta > 0$

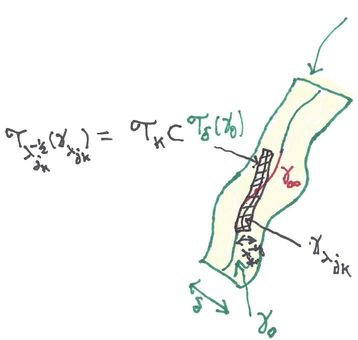
$$\int_{\mathcal{T}_\delta(\gamma_0)} |e_{\lambda_{j_k}}|^2 dV_g \not\to |\mathcal{T}_\delta(\gamma_0)|/|M| \approx \delta. \tag{6}$$

Assumptions $\implies$ $\exists$ subsequence of e.v.'s s.t. $\|e_{\lambda_{j_k}}\|_4 \ge c_1 \lambda_{j_k}^{1/8}$.
By CS 2011 KN theorem $\implies$ $\exists\, \gamma_{\lambda_{j_k}} \in \varPi$ and $\lambda_{j_k}^{-1/2}$-tubes $\mathcal{T}_{j_k}$ about these and a constant $c_0 > 0$ so that

$$\int_{\mathcal{T}_{j_k}} |e_{\lambda_{j_k}}|^2 dV_g \ge c_0.$$

$\varPi \approx M \times S^1$ compact $\implies$ after passing to further subsequence can assume $\gamma_{\lambda_{j_k}} \to \gamma_\infty \in \varPi$. If γ_0 geod of length 2 containing γ_∞ w/ same center, have $\mathcal{T}_{j_k} \subset \mathcal{T}_\delta(\gamma_0)$, k large $\implies$ (6) if $\delta \ll c_0$.

Shrinking tubes argument for scarring



$$c_0 \le \int_{\mathcal{T}_{\lambda_{jk}^{-1/2}}} |e_{\lambda_{jk}}|^2 \, dV_g \le \int_{\mathcal{T}_\delta(\gamma_0)} |e_{\lambda_{jk}}|^2 \, dV_g$$

$$\not\lesssim |\mathcal{T}_\delta(\gamma_0)| \approx \delta,$$

if $\delta \ll c_0$

2) Nonpos curv: Small L^4-norms & **no extreme scarring**

Theorem 3.(CS-Zelditch). If (M, g) has nonpositive curvature then

$$\sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |e_\lambda|^2 \, dV_g = o(1). \qquad (7)$$

By (7) and CS theorem, know that for manifolds with nonpositive curvature we have $\|e_\lambda\|_{L^4(M)} = o(\lambda^{\frac{1}{8}})$, and also the restriction estimates $(\int_\gamma |e_\lambda|^2 ds)^{1/2} = o(\lambda^{1/4})$, $\gamma \in \Pi$, improving on Burq-Gérard-Tzvetkov in this case. Recall blue estimates are equivalent when $n = 2$.

X. Chen and CS: stronger restriction ests $(\int_\gamma |e_\lambda|^4 \, ds)^{1/4} = o(\lambda^{1/4})$. We shall sketch proof of (7) using proof of XC-CS.

M. Blair and CS generalized (7) to higher dimensions and obtained o-L^p estimates too for nonpositive curvature. Turns out that (*unlike 2-d*), **cannot** use restriction estimates for latter; only L^2-bounds for tubes work. *Restriction estimates too singular.*

Setting up proof of o-restriction estimates nonpos curv

Recall we have (7) $\iff$ $\sup_{\gamma\in\Pi}\|e_\lambda\|_{L^2(\gamma)}=o(\lambda^{\frac14})$

Recall $\rho(T(\lambda-\sqrt{-\Delta_g}))e_\lambda=e_\lambda$ if $\rho(0)=1$. So have above if

$\|\rho(T(\lambda-\sqrt{-\Delta_g}))\|_{L^2(M)\to L^2(\gamma)}\le C\lambda^{\frac14}/T^{\frac14}$, λ large. $\iff$

$$\|\chi(T(\lambda-\sqrt{-\Delta_g}))\|_{L^2(\gamma)\to L^2(\gamma)}\le C\lambda^{1/2}/\sqrt{T},\ \chi=|\rho|^2,\ \lambda\ge\Lambda(T).$$

We may assume $\rho\in\mathcal{S}$ is even and $\hat\rho(t)=0,\ |t|\ge 1/2$. Then $\chi\in\mathcal{S}$ is even and $\hat\chi(t)=0$, $|t|>1$ and so

$$\chi(T(\lambda-\sqrt{-\Delta_g}))=\frac{1}{\pi T}\int_{-T}^{T}\hat\chi(t/T)e^{it\lambda}\cos(t\sqrt{-\Delta_g})\,dt+O(\lambda^{-N}).$$

Hadamard's miracles

By Cartan-Hadamard theorem, given any point $x_0 \in M$, the map

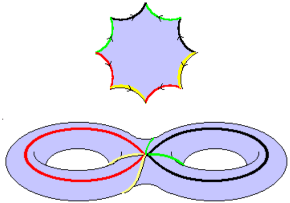
$$\kappa = \exp_{x_0} : TM \cong \mathbb{R}^2 \to M$$

is covering map. (**Hadamard 1898** $n = 2$)
Therefore, we have the close cousin of the classical *Poisson summation formula*:

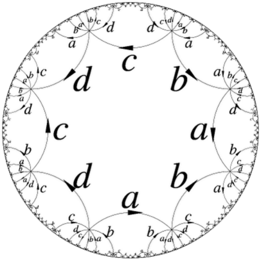
$$\cos(t\sqrt{-\Delta_g})(x, y) = \sum_{\alpha \in \Gamma} \cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y}))$$

- $\Gamma : \mathbb{R}^2 \to \mathbb{R}^2$ *deck transformations* (i.e., group of diffeomorphisms s.t. $\kappa \circ \alpha = \kappa$)
- $M \cong \mathbb{R}^2/\Gamma$
- Here, if $D \subset \mathbb{R}^2$ fund. domain for M, identify $x \in M$ w/ $\tilde{x} \in D$.
- $\tilde{g} = \kappa^* g$ (pullback of metric g on M via covering map)
- **Last miracle:** Can compute $\cos t\sqrt{-\Delta_{\tilde{g}}}$ using **Hadamard parametrix (1923)** as $(\mathbb{R}^2, \tilde{g})$ no conjugate points, by 1898 theorem.

Example: Double torus (constant curvature $= -1$)



Hyperbolic octagon is a fundamental domain for double torus



Translations of fundamental domain by deck transformations in Poincaré disk

Proof of o-restriction estimates using oscillatory integrals

Using Fourier transform formula for $\chi(T(\lambda - \sqrt{-\Delta_g}))$ and formula for $\cos t\sqrt{-\Delta_g}$, have desired $L^2(\gamma) \to L^2(\gamma)$ bounds for this restriction of this operator if for $\lambda \gg 1$

$$\Big(\int_{-1/2}^{1/2}\Big|\int_{-1/2}^{1/2} K(t,s)h(s)\,ds\Big|^2 dt\Big)^{1/2} \le C\frac{\lambda^{1/2}}{T^{\frac{1}{2}}}\Big(\int_{-1/2}^{1/2}|h(s)|^2 ds\Big)^{1/2}, \quad (8)$$

where if $\tilde{\gamma}(t)$, $|t| \le 1/2$ is lift of unit length geodesic $\gamma \in \Pi$,

$$\chi(T(\lambda - \sqrt{-\Delta_g}))(\gamma(t),\gamma(s)) = K(t,s) = \sum_{\alpha\in\Gamma} K_\alpha(t,s)$$

$$K_\alpha(t,s) = \frac{1}{\pi T}\int_{-T}^{T}\hat{\chi}(\tau/T)e^{i\lambda\tau}\big(\cos\tau\sqrt{-\Delta_{\tilde{g}}}\big)(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))d\tau.$$

Finite sum: (potentially $\exp(cT)$ terms!!) Since $K_\alpha = 0$ if $d_{\tilde{g}}(D,\alpha(D)) > T$. By **Hadamard parametrix**:

$$K_\alpha \approx T^{-1}\lambda^{1/2}e^{i\lambda d_{\tilde{g}}(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))}/(d_{\tilde{g}}(\tilde{\gamma}(t),\tilde{\alpha}(\gamma(s)))^{1/2} + l.o.t.$$

Let $T_\alpha : L^2([-1/2,1/2]) \to L^2([-1/2,1/2])$ integral operator with kernel K_α.

Proof, continued: Stabilizers and non-Stabilizers

Clearly $\|T_{Identity}\|_{L^2\to L^2} \le C\lambda^{\frac{1}{2}}/T$, as $d_{\tilde{g}}(\tilde{\gamma}(t),\tilde{\gamma}(s)) = |t-s|$.

$$T_\alpha h = \lambda^{\frac{1}{2}} T^{-1} \int_{-1/2}^{1/2} e^{i\lambda\phi_\alpha(t,s)} a_\alpha(t,s) h(s)\, ds, \quad \alpha \neq Identity,$$

smooth osc int op if $\phi_\alpha(t,s) = d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))$, $a_\alpha = 1/(\phi_\alpha)^{1/2}$.

If $\tilde{\gamma} = \{\tilde{\gamma}(t) : t \in \mathbb{R}\}$, then $\alpha(\tilde{\gamma})$ a geodesic (by Hadamard).

Two cases: i) $\alpha(\tilde{\gamma}) = \tilde{\gamma}$, or ii) $\alpha(\tilde{\gamma}) \neq \tilde{\gamma}$

Case i): $\gamma \in \varPi$ must be a unit segment of a periodic geodesic of period ℓ, and $\alpha(\tilde{\gamma}(s)) = \tilde{\gamma}(s+k\ell)$ for a unique $k \in \mathbb{Z}$. Then $\alpha \in \text{Stab}(\tilde{\gamma})$ and $\phi_\alpha \equiv |t-s-k\ell|$, so *trivial oscillators* **but** $|K_\alpha| \le Ck^{-1/2}$. Thus,

$$\sum_{\alpha\in\text{Stab }(\tilde{\gamma})} \|T_\alpha\|_{L^2\to L^2} \le CT^{-1}\lambda^{1/2}(1 + \sum_{0<k\le T} |k\ell|^{-1/2})) \le C\lambda^{1/2}T^{-1/2}.$$

Conclude that contribution of stabilizer group is as desired.

Suffices to show $\|T_\alpha\|_{L^2\to L^2} \le c_\alpha\lambda^{1/4},\ \alpha \notin \text{Stab }(\tilde{\gamma})$ (**smaller power of λ allows control of Huge Sum**).

Non-stabilizers and **oscillatory integrals (finally)**

$$T_\alpha h = \lambda^{\frac{1}{2}} T^{-1} \int_{-1/2}^{1/2} e^{i\lambda\phi_\alpha(t,s)} a_\alpha(t,s) h(s)\, ds, \quad \alpha(\tilde{\gamma}) \neq \tilde{\gamma}, \phi_\alpha = d_{\tilde{g}}(\tilde{\gamma}(t), \tilde{\alpha}(\gamma(s)))$$

By **Hadamard again:** either i) $\tilde{\gamma}$ and $\alpha(\tilde{\gamma})$ are *disjoint* or ii) they intersect at **unique** point $x_0(\alpha)$

By **Gauss:** In case i) always have $\partial_t \partial_s \phi_\alpha(t,s) \neq 0$ and so get $\| T_\alpha \|_{L^2 \to L^2} \leq c_\alpha$ in this case from **Hörmander's osc integral theorem**.

By **Gauss** again: In case ii) also have $\partial_t \partial_s \phi_\alpha(t,s) \neq 0$ if both $\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)) \neq x_0(\alpha)$, while if one equals $x_0(\alpha)$ have $|\partial_t^2 \partial_s \phi_\alpha(t,s)| + |\partial_t \partial_s^2 \phi_\alpha(t,s)| \neq 0$, and so by **oscillatory integral theorem of Greenleaf-Seeger and Phong-Stein** know $\| T_\alpha \|_{L^2 \to L^2} \leq c_\alpha \lambda^{1/4}$.

Period integrals (joint with Xuehua Chen)

By a similar argument, can show that if (M, g) has **strictly negative curvature** and $\gamma_{per} \in M$ is a periodic geodesic, have

$$\int_{\gamma_{per}} e_\lambda \, ds = o(1), \quad \lambda \to \infty \tag{9}$$

Using Kuznetsov trace formulae (1980) (*constant curvature*), **Good (1983)** and **Hejhal (1982)** showed above is $O(1)$.

Zelditch (1992) also obtained $O(1)$ bounds w/out curvature assumptions (and much more) using microlocal analysis.

Need to assume curvature is **negative**. For on $\mathbb{T}^2 \cong [-\pi, \pi)^2$ if $e_\lambda(x_1, x_2) = \cos \lambda x_1$, $\lambda \in \mathbb{Z}$ and $\gamma_{per} = (0, t)$, $t \in [-\pi, \pi)$, above integral $\equiv 2\pi$.

Miracle allowing you to control oscillatory integrals arising in proof of (9): If the curvature is **negative** then the sum of the angles for quadrilaterals is **strictly** less than 360°