Focal points and sup-norms of eigenfunctions

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Background

Compact *real analytic* manifold (M, g) of dimension $n \geq 2$.
Eigenfunctions:

$$-\Delta e_j(x) = \lambda_j^2 e_j(x), \qquad \int |e_j|^2 \, dV = 1$$

Give fundamental modes of vibration: $u_j(t,x) = \cos t\lambda_j \, e_j(x)$.
Know

$$\|e_j\|_{L^\infty(M)} \leq C\lambda_j^{\frac{n-1}{2}}$$

Question: When can you improve this estimate for eigenfunctions (or, more generally, quasi-modes)?

Since $u_j(t,x)$ provide high-frequency solutions of wave equations, $(\partial_t^2 - \Delta)u_j = 0$, *expect answer to depend on long-term dynamics of geodesic flow* (known to be linked to propagation of singularities for $\partial_t^2 - \Delta$)

Terminology to be used

Say that $\|e_j\|_{L^\infty(M)} = O(\lambda_j^{\frac{n-1}{2}})$, if, as above

$$\|e_j\|_{L^\infty(M)} \le C\lambda_j^{\frac{n-1}{2}}.$$

We are interested in when we have improvements, $\|e_j\|_{L^\infty(M)} = o(\lambda_j^{\frac{n-1}{2}})$, meaning that

$$\limsup_{\lambda_j\to\infty} \lambda_j^{-\frac{n-1}{2}} \|e_j\|_{L^\infty(M)} = 0.$$

If we cannot do this, we say that $\|e_j\|_{L^\infty(M)} = \Omega(\lambda_j^{\frac{n-1}{2}})$, meaning that

$$\limsup_{\lambda_j\to\infty} \lambda_j^{-\frac{n-1}{2}} \|e_j\|_{L^\infty(M)} > 0.$$

(We'll have to capture $\Omega(\lambda_j^{\frac{n-1}{2}})$ sup-norm behavior using *quasi-modes*, though, in which case we'll also be able to use *liminf* over natural sequence μ_k of "frequencies" arising from the geometry)

Loops through a point

Let $\Phi_t(x,\xi) = (x(t),\xi(t))$, with $(x(0),\xi(0)) = (x,\xi) \in S^*_x M$, denote unit-speed geodesic flow in cosphere bundle. Thus, $\gamma = \{x(t)\}$ is geodesic in M starting at x.

Let

$$\mathcal{L}_{x_0} = \{\xi \in S^*_{x_0}M : \ \Phi_t(x_0,\xi) = (x_0,\xi(t)), \text{ some } \ t > 0\}$$

denote initial directions of loops through x_0.

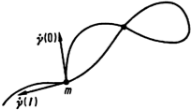


Fig. 7.6.1. A nonsimple nonclosed geodesic loop issuing from *m*

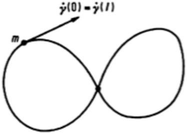


Fig. 7.6.2. A nonsimple closed geodesic issuing from *m*

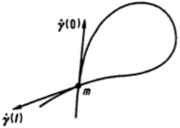


Fig. 7.6.3. A simple nonclosed geodesic loop issuing from *m*

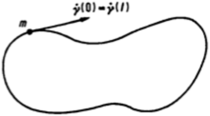


Fig. 7.6.4. A simply closed geodesic issuing from *m*

Figures from A. Besse, *Manifolds all of whose geodesics are closed*, (Springer, 1978).

Why *real analytic*?

In the *real analytic* case, can show that the set of looping directions satisfies either

$$|\mathcal{L}_{x_0}| = 0$$

or

$$\mathcal{L}_{x_0} = S^*_{x_0}M$$

AND there is an $\ell > 0$ so that $\Phi_\ell(x_0, \xi) = (x_0, \eta(\xi))$.
(i.e., all geodesics starting at x_0 loop back in time ℓ).

Say that x_0 a *self-focal point* [sfp] in second case

A Few Examples

- $\mathbb{T}^n$, then $|\mathcal{L}_x| = 0$ for all x. (No spf's)
- S^n, have $\Phi_{2\pi}(x,\xi) = (x,\xi)$, i.e., periodic flow (Everything a sfp)
- Surfaces of revolution. Periodic at poles

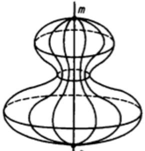
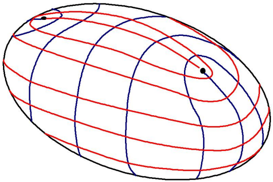


Fig. 0.3

- Triaxial ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, 0 < a < b < c < 1$. Loops of equal length thru 4 *umbilic points*



Latter has much different dynamics: Call these 4 points "dissipative"

Dissipative self-focal points $x_0 \in M$

If x_0 is a spf, there's a minimum $\ell > 0$ (first return time) so that $\Phi_\ell(x_0, \xi) = (x_0, \eta(\xi))\ \forall \xi \in S^*_{x_0}M$.
Call

$$\eta: S^*_{x_0}M \to S^*_{x_0}M \quad \textit{first return map}$$

Defines unitary operator (**Perron-Frobenius operator**)
$U = U_{x_0} : L^2(S^*_{x_0}M) \to L^2(S^*_{x_0}M)$

$$f \to Uf(\xi) = f(\eta(\xi))\sqrt{J(\xi)}, \quad J = \text{Jacobian}.$$

Define: sfp $x_0 \in M$ **dissipative** if U has no invariant L^2-functions, i.e., $0 \neq f \in L^2$ with $Uf = f$.

Calc III facts

U is unitary since

$$\int |Uf|^2 = \int |f(\eta(\xi)|^2 J(\xi) = \int |f|^2.$$

Integrals taken over S_x^*M with respect to Liouville measure (restriction of $d\xi dx$ measure)

Also, if $\mathbb{1} \equiv 1$ then $U\mathbb{1} = \sqrt{J(\xi)}$, and since $\langle f, g\rangle = \int f\,\overline{g}$, have

$$\int \sqrt{J(\xi)} = \langle U\mathbb{1}, \mathbb{1}\rangle$$

Similarly, by chain rule, if $\eta^\nu = \eta \circ \eta \circ \cdots \circ \eta$ (ν-times) and if J^ν is the Jacobian

$$\int \sqrt{J^\nu(\xi)} = \langle U^\nu \mathbb{1}, \mathbb{1}\rangle.$$

von Neumann Ergodic Theorem and self-focal points

Assume x is a self-focal point.
Then if $\mathbb{1} \equiv 1$ on S_x^*M and $\eta^\nu = \eta \circ \eta \circ \cdots \circ \eta$ (ν-times), if $J^\nu(\xi)$ is Jacobian and $d\mu$ is Liouville measure on S_x^*M, *by von Neumann*, as $T \to \infty$,

$$T^{-1} \sum_{\nu=1}^{T} \int_{S_x^*M} \sqrt{J^\nu(\xi)}\, d\mu(\xi) = \langle T^{-1} \sum_{\nu=1}^{T} U^\nu \mathbb{1},\, \mathbb{1} \rangle \to \langle \Pi(\mathbb{1}),\, \mathbb{1} \rangle,$$

where $\Pi : L^2(S_x^*M) \to L^2(S_x^*M)$ is projection onto U-invariant functions.
Conclude (as $U \geq 0$)

$$T^{-1} \sum_{\nu=1}^{T} \int_{S_x^*M} \sqrt{J^\nu(\xi)}\, d\mu(\xi) \longrightarrow \begin{cases} 0, & \text{if } x \text{ dissipative} \\ c_x > 0, & \text{if } x \text{ not dissipative} \end{cases}$$

Necessary and sufficient conditions for improved sup-norms

Define projection onto δ-bands of spectrum $(0 < \delta \leq 1)$ by

$$\chi_{[\lambda,\lambda+\delta]} f = \sum_{\lambda_j \in [\lambda,\lambda+\delta]} E_j f,$$

where $E_j f(x) = \langle f, e_j \rangle \, e_j(x)$ is proj onto j-th eigenspace.

Know $\|\chi_{[\lambda,\lambda+1]}\|_{L^2(M) \to L^\infty(M)} = O(\lambda^{\frac{n-1}{2}})$. *Local estimate—always sharp.*

Only possible improvements if use shrinking bands of intervals (i.e., $\delta = \delta(\lambda) \searrow 0$ as $\lambda \to \infty$).

Problem: $\|\chi_{[\lambda,\lambda+o(1)]}\|_{L^2 \to L^\infty} = o(\lambda^{\frac{n-1}{2}})$?

Means: $\forall \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ and $\Lambda_\varepsilon < \infty$ so that

$$\|\chi_{[\lambda,\lambda+\delta(\varepsilon)]} f\|_\infty \leq \varepsilon \lambda^{\frac{n-1}{2}} \|f\|_2, \quad \lambda \geq \Lambda_\varepsilon. \tag{1}$$

Main result

Theorem

Have improved sup-norm bounds, (1), if and only if every self-focal point is dissipative.

As we'll see there's an equivalent, but more natural formulation, when $n = 2$.

Since $\chi_{[\lambda,\lambda+\delta]} e_j = e_j$ if $\lambda_j \in [\lambda, \lambda + \delta]$, conclude that we have the following result for eigenfunctions:

Corollary

If all self-focal points are dissipative then

$$\|e_j\|_\infty = o(\lambda_j^{\frac{n-1}{2}}).$$

Some earlier results

- S-Zelditch (2002) Have (1) if $|\mathcal{L}_x| = 0$ for all $x \in M$.
- S-Toth Zelditch (2011). Same if $|\mathcal{R}_x| = 0$, where $\mathcal{R}_x \subset S_x^*M$ denotes the set of recurrent directions for the flow. Specifically initial unit directions ξ such that, given $\varepsilon > 0$, there is a t_0 so that $\Phi_{t_0}(x,\xi) \in S_x^*M$ (loops back) and $|\Phi_{t_0}(x,\xi) - (x,\xi)| < \varepsilon$.
- Note that if $\mathcal{C}_x \subset S_x^*M$ is the set of initial directions for periodic geodesics (i.e. smoothly closed loops) through x, then have

$$\mathcal{C}_x \subset \mathcal{R}_x \subset \mathcal{L}_x \text{ and so } |\mathcal{L}_x| = 0 \implies |\mathcal{R}_x| = 0 \implies |\mathcal{C}_x| = 0.$$

- By the Poincaré recurrence theorem if x is a sfp and $|\mathcal{R}_x| = 0$ then the Perron-Frobenius operator U_x cannot have $0 \neq f \in L^2(S_x^*M)$ with $U_x f = f$. For then $|f|^2 d\mu$ a finite invariant measure in the class of $d\mu$. Whence, by Poincaré, a.e. point with respect to $|f|^2 d\mu$ would be recurrent and so $|\mathcal{R}_x| > 0$.

Some earlier results, continued

Recall Weyl formula: $N(\lambda) = c_M \lambda^n + O(\lambda^{n-1})$, if $N(\lambda) = \#\lambda_j \le \lambda$.

- By the Duistermaat-Guillemin Theorem (1975), *if the set $\mathcal{C} \subset S^*M$ of all periodic points for the geodesic flow is of measure zero, then the error term in the Weyl formula is $o(\lambda^{n-1})$.*
- For many years thought that $|\mathcal{C}_x| = 0$ for all $x \in M$, should be sufficient for the $o(\lambda^{\frac{n-1}{2}})$ sup-norm bounds (1). *Wrong* (?)
- On the other hand, as we'll see below, if there is a non-dissipative sfp $x_0 \in M$ then the recipe for producing quasi-modes that are large there is related to another theorem in D-G ('75).

Equivalent *quasi-mode formulation* of Main Theorem

Functions $\phi_{\lambda_k} \in C^\infty(M)$ are quasi-modes of order $o(\lambda)$ ("*fat quasi-modes*") if

$$\int_M |\phi_{\lambda_k}|^2 \, dV = 1 \text{ and } \|(\Delta + \lambda_k^2)\phi_{\lambda_k}\|_2 = o(\lambda_k). \tag{2}$$

(N.B. as $|\lambda_k^2 - \lambda_j^2| = |\lambda_k - \lambda_j| \cdot (\lambda_k + \lambda_j)$ measures L^2-concentration in $o(1)$-sized intervals)

Corollary

Have

$$\|\phi_{\lambda_k}\|_\infty = o(\lambda_k^{\frac{n-1}{2}})$$

for every sequence $\lambda_k \to \infty$ if and only if every self-focal point is dissipative.

$\Omega(\lambda^{\frac{n-1}{2}})$ bounds at a non-dissipative sfp $x_0 \in M$

If x_0 is a non-dissipative sfp, let

$$\mu_k = \frac{2\pi}{\ell}(k + \beta/4), \quad k = 1, 2, \ldots, \tag{3}$$

where β is number of conjugate points (w/ multiplicity) along a unit speed geodesic $\{\gamma(t) : 0 \le t < \ell\}$, where ℓ is first return time and $\gamma(0) = \gamma(\ell) = x_0$

Then if $T_k \to \infty$ (slow enough) the "coherent state" at x_0

$$\phi_{\mu_k}(y) = \mu_k^{-\frac{n-1}{2}} \sum_{j=0}^{\infty} \rho(T_k(\mu_k - \lambda_j))\, e_j(x_0) e_j(y)$$

satisfies (2), and has $\phi_{\mu_k}(x_0) \approx \mu_k^{\frac{n-1}{2}}$, assuming that

$$0 \le \rho \in \mathcal{S}(\mathbb{R}) \text{ has } \rho(t) \ge 1, \ |t| \le 1, \ \hat{\rho} \ge 0, \text{ and } \operatorname{supp} \hat{\rho} \subset [-1, 1].$$

Related earlier result of Duistermaat-Guillemin (1975) and Weinstein (1977)

Theorem (Duistermaat-Guillemin/Weinstein)

If geodesic flow on M is periodic with minimal period $\ell > 0$ and μ_k as in (3) and intervals I_k are given by

$$I_k = [\mu_k - Ck^{-1}, \mu_k + Ck^{-1}],$$

with $C > 1$ sufficiently large but fixed, then

$$\text{Spectrum } \sqrt{-\Delta_g} \subset \bigcup I_k.$$

Ideas in our proofs

Want to show that if all sfp's are *dissipative* then, given $\varepsilon > 0$, if $T(\varepsilon)$, $\Lambda(\varepsilon)$ large

$$\|\chi_{[\lambda,\lambda+T^{-1}(\varepsilon)]}\|_{L^2\to L^\infty} \le \varepsilon\lambda^{\frac{n-1}{2}}, \quad \lambda \ge \Lambda(\varepsilon),$$

or this cannot happen if there are non-dissipative sftp's.

Concentrate for now on former. By orthogonality,

$$\|\chi_{[\lambda,\lambda+T^{-1}(\varepsilon)]}\|^2_{L^2\to L^\infty} = \sup_{x\in M} \sum_{\lambda_j\in[\lambda,\lambda+T^{-1}(\varepsilon)]} \big(e_j(x)\big)^2,$$

and so task is equivalent to showing that

$$\sum_{\lambda_j\in[\lambda,\lambda+T^{-1}(\varepsilon)]} \big(e_j(x)\big)^2 \le \varepsilon^2\lambda^{n-1}, \quad \lambda \gg 1 \tag{4}$$

By earlier results of S-Zelditch, know have uniform such bounds near any point $x_0 \in M$ with $|\mathcal{L}_{x_0}| = 0$. So need same near given dissipative sfp.

o-bounds near dissipative self-focal point $x_0 \in M$

Use above bump function $0 \le \rho \in \mathcal{S}$ satisfying $\rho(t) \ge 1,\ t \in [-1,1]$ and supp $\hat\rho \subset [-1,1]$ and $\hat\rho \ge 0$. Have

$$\sum_{\lambda_j \in [\lambda, \lambda + T^{-1}(\varepsilon)]} \big(e_j(x)\big)^2 \le \sum_{j=0}^{\infty} \rho(T(\lambda - \lambda_j))\,(e_j(x))^2, \quad T = T(\varepsilon),$$

and so would have (4) near our diss sfp x_0 if could show there that

$$\sum_{j=0}^{\infty} \rho(T(\lambda - \lambda_j))\,(e_j(x))^2 \le \varepsilon^2 \lambda^{n-1} \tag{5}$$

Use

$$\sum_{j=0}^{\infty} \rho(T(\lambda - \lambda_j))\,(e_j(x))^2 = \frac{1}{2\pi T} \int \hat\rho(t/T) e^{it\lambda} \sum_j \big(e^{-it\lambda_j}(e_j(x))^2\big)\, dt$$

$$= \frac{1}{2\pi T} \int_{-T}^{T} \hat\rho(t/T) e^{it\lambda}\, \big(e^{-it\sqrt{-\Delta}}\big)(x,x)\, dt$$

FIOs and propagation of singularities

Near any t_0 (mod C^∞) can write $\exp(-it\sqrt{-\Delta})(x,y)$ as finite sum $j = 1, \dots, N(t_0)$ of terms

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{iS_j(t,y,\xi) - ix\cdot\xi} a_j(t,x,y,\xi)\, d\xi,$$

where

$$a \in S^0, \quad a(\ell\nu, x, x, \xi) = e^{-i\nu\beta\pi/2}\sqrt{J^\nu(\xi)}, \quad \nu \in \mathbb{Z},$$

$$\partial_t S_j = p(y, \nabla_x S_j)$$

Factors $i^{-\nu\beta}$ called *Maslov factors*

Our assumptions imply that $\exp(-it\sqrt{-\Delta})(y,y)$ is smooth if y close to sfp x and $t \in \operatorname{supp} \hat{\rho}(\,\cdot\,/T)$ is not close to

$$\{\ell\nu : \nu = 0, \pm 1, \pm 2, \dots\} \cap [-T, T]$$

Punch line via stationary phase

Using above facts write $\xi = r\omega$, $\omega \in S_x^*M$ and *use stationary phase* in t, r variables. **Miracle:** *Modulo lower order terms in* λ, LHS of (5) is

$$\sum \rho(T(\lambda-\lambda_j))(e_j(x))^2$$
$$= \lambda^{n-1} \sum_{|\nu|\le T} \big(e^{i\ell\nu\lambda} e^{-i\nu\beta\pi/2}\big)\hat{\rho}(\ell\nu/T)\,\Big(\frac{1}{T}\int_{S_x^*M} \sqrt{J^\nu(\omega)}\,d\mu(\omega)\Big) \quad (6)$$

If $x = x_0$ *dissipative sfp* then (mod l.o.t.'s) this is

$$\le C\lambda^{n-1} T^{-1} \sum_{|\nu|\le T} \langle U^\nu \mathbb{1}, \mathbb{1}\rangle,$$

and coefficient is small if T large by the von Neumann ergodic Theorem

If x *non-diss sfp* and $\lambda = \mu_k$ as in (3) then blue factor in (6) is one and $\hat{\rho} \ge 0$. Thus, if $0 < c_0 = \lim_{T\to\infty} T^{-1} \sum_{|\nu|\le T} \langle U^\nu \mathbb{1}, \mathbb{1}\rangle$, the LHS of (6) $\approx c_0 \mu_k^{n-1}$ for every fixed $T > 1$ if $\lambda \gg 1$

More clarity in 2-dimensions

Using the time-reversibility of geodesic flow and elementary facts about circle maps and dynamics, see that when $n = 2$ *if there is a non-dissipative self-focal point* $x \in M$ then the first return map $\eta : S_x^* M \to S_x^* M$ must satisfy

$$\eta \circ \eta = \mathit{Identity}$$

Thus, if ℓ is the first return time for the flow at x either all geodesics emanating there are periodic with period ℓ or 2ℓ. Therefore, we have:

Theorem

If (M, g) is a 2-dimensional Riemannian manifold then $\|\phi_{\lambda_k}\|_\infty = o(\lambda_k^{(n-1)/2})$ for every sequence $\lambda_k \to \infty$ of frequencies and corresponding quasimodes of order $o(\lambda)$ if and only if there is no point through which the geodesic flow is periodic.

Remarks and questions

Did not use real analyticity in essential way. For instance, get $\Omega(\mu_k^{\frac{n-1}{2}})$-sized quasi-modes at any $x \in M$ if all geodesics through x loop back at some time $\ell > 0$, and $\exists f \in L^2$, $Uf = f$, $f \neq 0$

As in the real analytic case above, the q.m.'s satisfy $\phi_{\mu_k}(x) = \Omega(\mu_k^{\frac{n-1}{2}})$ and

$$\int_M |\phi_{\mu_k}|^2 \, dV = 1 \text{ and } \|(\Delta + \mu_k^2)\phi_{\mu_k}\|_2 = o(\mu_k).$$

Question: *Can you reach a stronger conclusion by replacing the red term by* $O(1)$*?* (More traditional definition of q.m.'s.)

If the Perron-Frobenius operator $Uf(\xi) = \sqrt{J(\xi)}\, f(\eta(\xi))$ had a *smooth invariant function*, would be able to solve a transport equation and do this. By above, can do this when $n = 2$. (Either $f = \mathbb{1}$ or $f = U\mathbb{1}$ works.)

When $n \geq 3$, the conclusion or assumption just concerns L^2-invariant functions (used to obtain o-bounds via von Neumann when all sfp's diss).

Remarks and questions, continued

- By interpolation with CS '88 L^p estimates: Know that if all sfp's are dissipative then every quasi-mode as in (2) must satisfy

$$\|\phi_\lambda\|_{L^p(M)} = o(\lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}), \quad p > \frac{2(n+1)}{n-1} \tag{7}$$

- What about *endpoint* $p = \frac{2(n+1)}{n-1}$?
- Can show that all sfp's being dissipative is a *necessary condition* for (7) when $p = \frac{2(n+1)}{n-1}$.
- Much recent work (Bourgain, CS, CS-Zelditch, Blair-CS) on obtaining improved L^p estimates when $2 < p < \frac{2(n+1)}{n-1}$. Relevant thing for this range: *concentration along periodic geodesics* (as opposed to concentration at points).