L^p-resolvent estimates for compact Riemannian manifolds and hyperbolic space

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I.) Basic problems

Interested in uniform estimates of the form

$$\|u\|_{L^s(M)} \le C\|(\Delta+\zeta)u\|_{L^r(M)}, \quad u \in C_0^\infty,\ \zeta \in \mathcal{R} \subset \mathbb{C}$$

- Δ=Laplacian on (M,g), norms taken w.r.t. volume element
- (M,g) either compact w/out boundary, or simply connected with constant curvature (e.g., $\mathbb{H}^n$)
- M compact $\zeta = \lambda + i$ for appropriate $C = C(\lambda) = \lambda^{\sigma(r,s)}$ implies bounds for eigenfunctions
- M noncompact and $r = s = 2$, cutoff resolvent bounds for $(\Delta+\zeta)^{-1}$ important in semiclassical analysis (resonances) and sensitive to *trapping* of geodesics—distribution of resonances.
- Applications to control theory.
- For L^p-resolvent bounds and M compact or noncompact: *Expect estimates to be related to spectrum of Δ and dynamics of geodesic flow*

II a.) Optimal results for Euclidean space

Suppose that

- $n(\frac{1}{r} - \frac{1}{s}) = 2$
- and $\min\left(|\frac{1}{r} - \frac{1}{2}|, |\frac{1}{s} - \frac{1}{2}| \right) > \frac{1}{2n}$, (i.e., $\frac{2n}{n-1} < s < \frac{2n}{n-3}$)

Kenig, Ruiz, S. ('87):

If $n \geq 3$ and r, s as above $\exists C_{r,s}$ so that

$$\|u\|_{L^s(\mathbb{R}^n)} \leq C_{r,s} \|(\Delta + \zeta)u\|_{L^r(\mathbb{R}^n)}, \quad \zeta \in \mathbb{C},\ u \in C_0^\infty. \qquad (1)$$

Idea: If $f = (\Delta + \zeta)u$, then

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\hat{f}(\xi)}{\zeta - |\xi|^2}\, d\xi,$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\, dy, \quad f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

II b.) Euclidean restriction estimates

Worse case: $\zeta = 1 - i\varepsilon$, $\varepsilon \searrow 0$, since

$$\text{Im } (1 - i\varepsilon - |\xi|^2)^{-1} \to \pi \, dS, (\text{surface measure on } S^{n-1})$$

So (1) implies that

$$\| \int_{S^{n-1}} e^{ix\cdot\omega} \hat{f}(\omega) \, dS(\omega) \|_{L^s(\mathbb{R}^n)} \le C \|f\|_{L^r(\mathbb{R}^n)}, \qquad (2)$$

- See that you need $s > \frac{2n}{n-1}$ if $f \in \mathcal{S}$ with $\hat{f} = 1$ near S^{n-1}
- Case where $s' = r = \frac{2n}{n-2} \iff$ to special case of Stein-Tomas:

$$\Big(\int_{S^{n-1}} |\hat{u}|^2 \, dS \Big)^{\frac{1}{2}} \le C \|u\|_{L^p(\mathbb{R}^n)}, \quad 1 \le p \le \frac{2(n+1)}{n+3}. \qquad (3)$$

KRS: Can reverse argument that (1) $\implies$ (2) as well.

Remarks

If dE_λ, $\lambda \in [0,\infty)$ denotes the spectral measure for $\sqrt{-\Delta_{\mathbb{R}^n}}$, then, after rescaling, (2) is equivalent to

$$\|dE_\lambda\|_{L^r(\mathbb{R}^n)\to L^s(\mathbb{R}^n)} = \lim_{\varepsilon\searrow 0} \varepsilon^{-1} \|\mathbb{1}_{[\lambda,\lambda+\varepsilon]}(\sqrt{-\Delta_{\mathbb{R}^n}})\|_{L^r\to L^s} \le C\lambda. \quad (4)$$

- Continuous spectrum of $\Delta_{\mathbb{R}^n}$ responsible for this as well as resolvent Euclidean estimates (1).
- Expect *different story for compact manifolds*, and, there, expect dynamics of geodesic flow and spectral properties of Δ_g to dictate how close you can come to (1) and (4)
- Expect also: Resolvent bounds corresponding to high frequencies can see spectrum and dynamics. Known to be true in many cases for L^2—also for L^p bounds.
- What about $\mathbb{H}^n$?

III.) Compact manifolds: Classical Spectral Theory

In 1910 **Sommerfeld**, followed 3 months later by **Lorentz**, gave famous lectures inspiring Weyl's work. Sommerfeld interested in forced vibration problem in dimensions $n = 1, 2, 3$: $(\Delta + \zeta)u(x) = f(x)$, $x \in \Omega \Subset \mathbb{R}^n$, $u|_{\partial\Omega} = 0$

Asked how properties of solution operator $(\Delta + \zeta)^{-1}$ related to solutions of the free vibration problem

$$(\Delta + \lambda_j^2)e_j(x) = 0, \;\; e_j|_{\partial\Omega} = 0, \quad \int_\Omega |e_j|^2 dx = 1$$

Kernel of solution operator: $\mathfrak{G}(x, y) = \sum_j \frac{e_j(x)e_j(y)}{\zeta - \lambda_j^2}$

Sommerfeld reasoned that

$(\Delta + \zeta)\mathfrak{G}(x, y) = \sum e_j(x)e_j(y)$ is "**spike function**"

Also conjectured that properties of $\mathfrak{G}(x, y)$ should be related to distribution of eigenvalues $\{\lambda_j\}$, and "cancellation from numerator" (oscillation of e.f.'s)

Weyl Law

Lorentz's subsequent 1910 lecture spelled out the eigenvalue problem more precisely and asked whether for the eigenvalues for the Dirichlet Laplacian in smooth domains $\Omega \subset \mathbb{R}^n$ one has for $N(\lambda) = \text{number } \lambda_j \leq \lambda$

$$N(\lambda) = (2\pi)^{-n}(\text{Vol } B)(\text{Vol } \Omega)\lambda^n + o(\lambda^n)$$

Hilbert: **No way in my lifetime**

Weyl: Yes! (several proofs in 1911-12 comparison arguments, heat kernel, Tauberian arguments...)

Sharp Weyl formula (Avakumovic '50s):

$$N(\lambda) = c_M \lambda^n + O(\lambda^{n-1}).$$

Improvements over the years in many special cases with "good" dynamics

IV.) Resolvent bounds for compact manifolds

Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$, interested in regions $\mathcal{R}(g)$ for which one can have uniform resolvent estimates:

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C\|(\Delta_g + \zeta)u\|_{L^{\frac{2n}{n+2}}(M)}, \ \zeta \in \mathcal{R}(g), \ u \in C^\infty.$$

Z. Shen (2001): For the torus $\mathbb{T}^{\mathbf{n}} = \mathbb{R}^{\mathbf{n}}/\mathbb{Z}^{\mathbf{n}}$ can take region to be

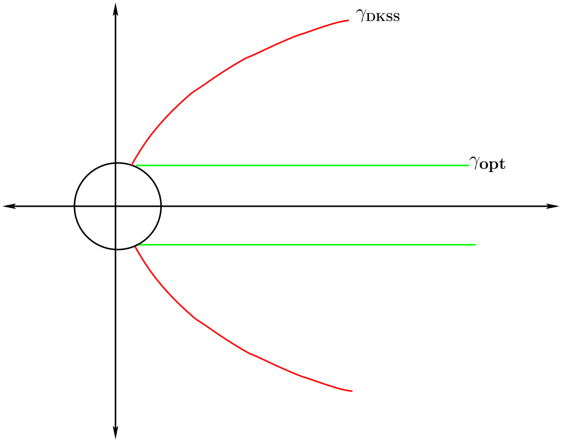
$$\{\zeta \in \mathbb{C} : \ \mathrm{Re}\, \zeta \leq (\mathrm{Im}\, \zeta)^2, \ |\zeta| \geq 1\}$$

Dos Santos Ferreira, Kenig and Salo [DKS] (2011): Same results for *any* compact Riemannian manifold

Problem raised by DKS for $(\Delta_g + \zeta)^{-1} : L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$

Does the DKS-S theorem hold for a larger region, specifically the region outside the curve γ_{opt}, which is $|\text{Im } \zeta| = 1$ (unit distance from spectrum of $-\Delta_g$)?

This would be natural Riemannian version of KRS results for $\mathbb{R}^n$.



V.) Weyl law & answer (w Bourgain, Shao & Yao [BSSY])

The answer is **NO:** *In some cases cannot come close to* γ_{opt} *and earlier bounds of DKS in fact cannot be improved:*
Write $\zeta = (\lambda + i\varepsilon(\lambda))^2 = \lambda^2 + 2i\varepsilon\lambda - \varepsilon^2$.

Then $\varepsilon(\lambda) = 1$ corresponds to γ_{DKSS} curve. (BSSY): If

$(\Delta + (\lambda + i\varepsilon(\lambda))^2)^{-1} : L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ uniformly, then

$$\begin{aligned} &\#\{\lambda_j : \lambda_j \in [\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)]\} \\ &\quad \le \text{Vol } \{\xi \in \mathbb{R}^n : |\xi| \in [\lambda - \varepsilon(\lambda), \lambda + \varepsilon(\lambda)]\} \lesssim \varepsilon(\lambda)\lambda^{n-1} \\ &\qquad = \text{Volume of } \varepsilon(\lambda) - \text{annulus about } \lambda S^{n-1}. \end{aligned}$$

Cannot hold if $\varepsilon(\lambda) \searrow 0$ for S^n as distinct eigenvalues are $\lambda = \sqrt{k(k+n-1)}$, repeating with multiplicity $\approx \lambda^{n-1}$.
γ_{opt} corresponds to $e(\lambda) = \lambda^{-1}$. DIFFICULT!!

Sommerfeld correct: Resolvent operators sensitive to spectral properties of Δ_g

Some cases of nonclustering spectrum

Some manifolds (M, g) where it is known that $\exists\, \varepsilon(\lambda) \searrow 0$ so that

$$\#\{\lambda_j : |\lambda_j - \lambda| \leq \varepsilon(\lambda)\} = O(\varepsilon(\lambda)\lambda^{n-1}),$$

which implies the above nonclustering condition for L^p-resolvent bounds.

Specifically:

- Manifolds of nonpositive curvature (**Bérard** '78): $\varepsilon(\lambda) = 1/\log\lambda$
- Standard n-torus, $\mathbb{T}^n$ (**Hlawka** '50): $\varepsilon(\lambda) = \lambda^{-\sigma_n}$, $\sigma_n = -1 + \frac{2}{n+1}$
- **Duistermaat-Guillemin (&Ivrii)** ('75) $\varepsilon(\lambda) = o(1)$ if (M, g) has zero measure of periodic geodesics (is a generic condition).

BSSY: Improvements of DKS-S in first two cases

Unlike the situation for the *n*-sphere, we can improve the earlier estimates of DKSS somewhat for i) manifolds of nonpositive curvature, and a bit more for ii) $\mathbb{T}^n$:

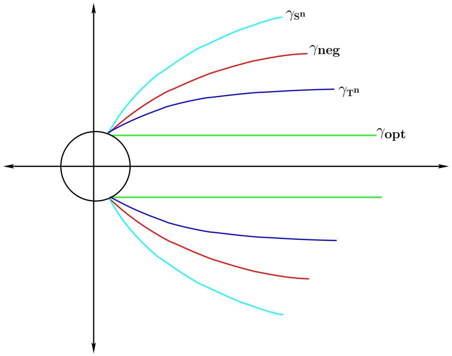


Figure : Various regions $\mathcal{R}(M, g)$ for $(\Delta_g + \zeta)^{-1}$

Improved bounds for numerator in Sommerfeld-Green fnct

Theorem (BSSY):

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \le C\big\|(\Delta_g + (\lambda + i\varepsilon(\lambda))^2)u\big\|_{L^{\frac{2n}{n+2}}(M)}, \tag{5}$$

if and only if

$$\Big\| \sum_{|\lambda-\lambda_j|\le\varepsilon(\lambda)} E_j f \Big\|_{L^{\frac{2n}{n-2}}(M)} \le C\lambda\varepsilon(\lambda)\|f\|_{L^{\frac{2n}{n+2}}(M)}. \tag{6}$$

- $E_j f = < f, e_j > \; e_j(x)$ projection of f onto j-th eigenspace. So above operator is proj of f onto $\varepsilon(\lambda)$-spectral band about λ.
- CS ('88) earlier proved this estimate with $\varepsilon(\lambda) = 1$ (sharp for sphere).
- Natural variant of (4), ie., Euclidean $L^r \to L^s, s' = r$ bounds for

$$\mathbb{1}_{[\lambda,\lambda+\varepsilon(\lambda)]}(\sqrt{-\Delta_{\mathbb{R}^n}})g = \int_{\{\xi\in\mathbb{R}^n:\,|\,|\xi|-\lambda|\le\varepsilon(\lambda)\}} e^{ix\cdot\xi}\,\hat{g}(\xi)\,d\xi$$

Spectral projection bds $\implies$ resolvent bds

Use the following variant of F.t. of Poisson summation kernel: If $P = \sqrt{-\Delta_g}$, $\zeta = (\lambda + i\mu)^2$,

$$\left(\Delta_g + (\lambda + i\mu)^2\right)^{-1} = \frac{\text{sgn } \mu}{i(\lambda + i\mu)} \int_0^\infty e^{i(\text{sgn } \mu)\lambda t} e^{-|\mu| t} (\cos tP)\, dt$$
$$= \ldots \int_0^\infty \beta(t) \ldots dt + \ldots \int_0^\infty (1 - \beta(t)) \ldots dt,$$

with $C_0^\infty \ni \beta = 1$ near origin.

- If β small support, 1st term uniformly bounded for all ζ (local piece). Use Hadamard parametrix and Stein's osc int theorem.
- 2nd term is multiplier operator $m_{\lambda,\mu}(P)$ with

  $$m_{\lambda,\mu}(\tau) = O((1 + |\lambda - \tau|)^{-N}) + O(|\mu|^{-1}(1 + |\mu|^{-1}|\lambda - \tau|)^{-N})$$

  and so can use old unit-band and improved $\mu = \varepsilon(\lambda)$-band bonds (and reverse TT^* argument) to handle it.

Improved spectral projection bounds of BSSY

Have

$$\Big\| \sum_{|\lambda-\lambda_j|\le\varepsilon(\lambda)} E_j f \Big\|_{L^{\frac{2n}{n-2}}(M)} \le C\lambda\varepsilon(\lambda)\|f\|_{L^{\frac{2n}{n+2}}(M)}.$$

with

- $\varepsilon(\lambda) = 1/\log\lambda$ if (M,g) has nonpositive curvature.
- For $\mathbb{T}^n$ if $\varepsilon(\lambda) = \lambda^{-\sigma_n}$ with

$$\sigma_n = \begin{cases} \frac{85}{252} \approx 0.337, \ \ n=3 \\ \frac{2(n-1)}{n^2+2n-2}, \ \ n\ge 4, \text{ even} \\ \frac{2(n-1)}{n(n+1)}, \ \ n\ge 5, \ \text{odd}. \end{cases}$$

- **Tools:** For $\mathbb{T}^n$ use Poisson summation formula and recent harmonic techniques developed by *Bennett-Carbery-Tao, Bourgain-Guth...*
  For nonpos curv use *Cartan-Hadamard thm* to lift calculation up to universal cover & variant of Poisson summation

Results w/Shanglin Huang for hyperbolic space

$\mathbb{R}^n$ w/ metric of const curv $-\kappa$, $\kappa > 0$: In geod polar coords

$$\Delta_{-\kappa} = \partial_r^2 + (n-1)\sqrt{\kappa}\coth(\sqrt{\kappa}r)\partial_r + (\sqrt{\kappa}\operatorname{csch}(\sqrt{\kappa}r)^2\Delta_{S^{n-1}},$$

$$dV_{-\kappa} = \Big(\frac{\sinh\sqrt{\kappa}r}{\sqrt{\kappa}}\Big)^{n-1} drd\theta, \quad 0 < r < \infty, \ -\kappa < 0,$$

Theorem

Let $n \geq 3$. Then for every r, s as above $\exists C_{r,s}$ so that $(-\kappa = -1)$ if $u \in C_0^\infty$

$$\|u\|_{L^s(\mathbb{H}^n, dV_{\mathbb{H}^n})} \leq C_{r,s}\big\|\big((\Delta_{\mathbb{H}^n} + (\tfrac{n-1}{2})^2) + \zeta\big)u\big\|_{L^r(\mathbb{H}^n, dV_{\mathbb{H}^n})}, \ |\zeta| \geq 1.$$

Also, have uniform bounds (indep of $-\kappa < 0$)

$$\|u\|_{L^s(\mathbb{R}^n, dV_{-k})} \leq C_{r,s}\big\|\big((\Delta_{-\kappa} + \kappa(\tfrac{n-1}{2})^2) + \zeta\big)u\big\|_{L^r(\mathbb{R}^n, dV_{-\kappa})}, \ |\zeta| \geq \kappa.$$

Letting $\kappa \to 0_+$ recover Euclidean estimates of KRS.

Proof of resolvent estimates for constant neg curv

Use that for the shifted Laplacians $-\Delta_{-\kappa} - \kappa(\frac{n-1}{2})^2$ (having spectrum $[0, \infty)$), have the following variants of Euclidean estimates (4) for $\sqrt{-\Delta_{-\kappa} - \kappa(\frac{n-1}{2})^2}$:

$$\left\| \mathbb{1}_{[\lambda,\lambda+\varepsilon]}(P_\kappa) \right\|_{L^r(dV_{-\kappa} \to L^s(dV_{-\kappa})} \le C\varepsilon\lambda, \quad \lambda \ge \kappa. \qquad (7)$$

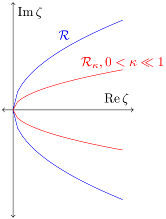
The condition that the spectral parameter be $\ge \kappa$ not necessary for $n = 3$.
You prove this using Stein's analytic interpolation argument used to prove Stein-Tomas and explicit formulae for functions of P_κ (coming from explicit formulae for fund solution of wave operators)

Use (7) to deal with large frequencies λ in spectral decomposition of resolvent operators

For low freq: Very handy to use Sobolev estimates for unshifted Laplacian: $(-\Delta_{\mathbb{H}^n})^{\alpha/2} : L^p(\alpha)(\mathbb{H}^n) \to L^{q(\alpha)}(\mathbb{H}^n)$ (exponents as in $\mathbb{R}^n$) from Cowling-Giulini-Meda ('93)

Constant positive curvature: round spheres

Also can prove continuous family of uniform estimates for spheres of constant curvature $\kappa > 0$ that imply the KRS estimates.



Do this by proving uniform $L^r \to L^s$ bounds for proj onto spherical harmonics of degree k. *Recover* $\mathbb{R}^n$ *bounds by letting* $\kappa \to 0_+$

Need $\|H_k\|_{L^r(S^n) \to L^s(S^n)}$ (proj onto spherical harmonics of deg k.)

For this, need to strengthen classical Darboux formula for asymptotics of projection kernel (i.e., "*zonal functions*" on S^n)

Use periodicity of wave group $t \to e^{it\sqrt{-\Delta_{S^n} + (\frac{n-1}{2})^2}}$.