MTAC lecture 5. 10/07/2015.

Bilinear Estimate $\sigma < h$.

$$\|p(x,hD)u\|_{L^2(M)} \lesssim h\|u\|_{L^2}. \quad \|p(x,\sigma D)v\|_{L^2(M)} \lesssim \sigma\|v\|_{L^2(M)}.$$

$$\|u\cdot v\|_{L^p} \lesssim \underbrace{G(h,\sigma)}_{?}\,\|u\|_{L^2}\,\|v\|_{L^2}.$$

Th^m. [Guo, Han, T]. p Laplace-like.

$$G(h,\sigma) = \begin{cases} h^{-\frac{1}{2}}\sigma^{-\frac{1}{2}+\frac{2}{p}} & 6 \le p \le \infty. \\ h^{-\frac{3}{4}+\frac{3}{2p}}\sigma^{-\frac{1}{4}+\frac{1}{2p}} & 3 \le p \le 6. \\ h^{-\frac{1}{4}}\sigma^{-\frac{1}{4}+\frac{1}{2p}} & 2 \le p \le 3. \end{cases}$$

Think of $\|u\cdot v\|_{L^p}$, not estimate $u(x)v(y)\big|_{\{x=y\}}$.

$L^1 \to L^\infty$: $\sigma^{-\alpha_\infty}h^{-\mu_\infty}(h+|t-s|)^{-\gamma_\infty}(\sigma+|t-s|)^{-\beta_\infty}$.

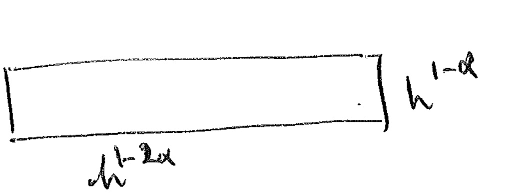
$L^2 \to L^2$: $\sigma^{-\alpha_2}h^{-\mu_2}(h+|t-s|)^{-\gamma_2}(\sigma+|t-s|)^{-\beta_2}$.

$|t-s| < \sigma$: high p.

$\sigma < |t-s| < h$: mid p.

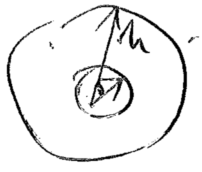
$h < |t-s|$: low p.

$T_h^\alpha = N_\alpha$ (at scale h).



$|T_h^\alpha| > c\,h^{-\frac{1}{2}+\alpha/2}$.

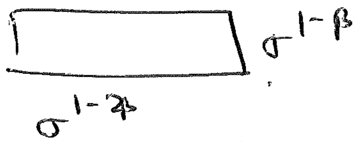
High p: $\alpha = 0$, $\beta = 0$.



So on $|x| < \varepsilon\sigma$, $|T^0_h| > ch^{-\frac12}$, $|T^0_\sigma| > c\sigma^{-\frac12}$.

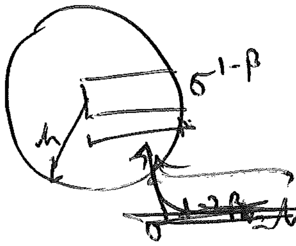
$$\|T^0_h T^0_\sigma\| > h^{-\frac12}\sigma^{-\frac12+\frac2p}.$$

T^β_σ:



$$|T^\beta_\sigma| > c\,\sigma^{-\frac12+\beta/2}$$

T^0_h:



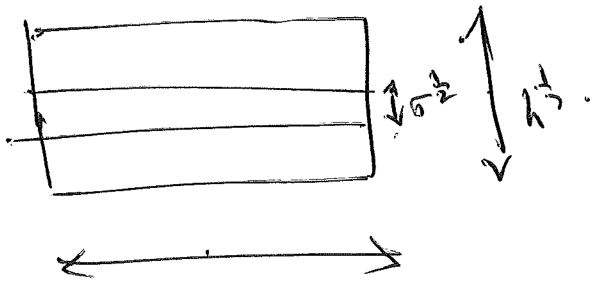
$$|T^0_h| > h^{-\frac12} \qquad |T^{\beta_h}_\sigma| > \sigma^{-\frac12+\frac{\beta_h}{2}}.$$

$(\sigma^{1-2\beta_h} = h)$ defining eq.

$$\|T^0_h T^{\beta_h}_\sigma\|_{L^p} \geq h^{-\frac12}\sigma^{-\frac12+\frac{\beta_h}{2}}\left(\sigma^{1-2\beta_h+1-\beta_h}\right)^{\frac1p}$$

$$= h^{-\frac34+\frac{3}{2p}}\sigma^{-\frac14+\frac{1}{2p}}$$

<u>here P:</u> $T^{h}_h\, T^{\frac12}_\sigma$.



$$\|T^{\frac12}_h T^{\frac12}_\sigma\| > h^{-\frac14}\sigma^{-\frac14}\cdot\sigma^{\frac{1}{2p}}.$$

$|T^{\frac12}_h| > h^{-\frac14}$, $|T^{\frac12}_\sigma| > \sigma^{-\frac14}$.

<u>Direction</u>. $u = \sum_i x_i(x, hD)u$, $v = \sum_j X_j(x,\sigma D)v$.

$$u\cdot v = \sum_{i,j} x_i(x,hD)u\; X_j(x,\sigma D)v.$$

Factorisation: $\exists\, \xi_i$ call it ξ_1.

$$|\partial_{\xi_1} p(x,\xi)| > c > 0 \quad \text{on spt } \chi_i.$$

Is $|\partial_{\xi_1} p(x,\xi)| > c > 0$ on spt χ_j

<u>Case 1</u>: Yes $\to$ propagation in same direction.

$$p(x,\xi) = e_1(x,\xi)(\xi_1 - a(x,\xi')) \quad \text{on } \chi_i \text{ spt}.$$
$$p(x,\xi) = e_2(x,\xi)(\xi_1 - a_2(x,\xi')) \quad \text{on } \chi_j \text{ spt}.$$

<u>Case 2</u>: ~~prop~~ No: propagation in diff directions.

$$p(x,\xi) = e_1(x,\xi)(\xi_1 - a(x,\xi')) \quad \text{on spt } \chi_1.$$
$$p(x,\xi) = e_2(x,\xi)(\xi_2 - a_2(x,\xi')) \quad \text{on spt } \chi_2.$$

Case 1: $x_1 = t$, $w(t,x,y) := u(t,x_2)\,v(t,y_2)$.

$$\left(hD_t - a_1(t, x_2, hD_{x_2}) - a_2(t, y_2, hD_{y_2})\right) w = O_{L^2}(h).$$

Hypersurface ~~est~~ $3D \to 2D$. $\underline{\underline{\sigma << h}}$ | Rest σ lines more oscillation and drag h along.

20 product $\|u_h(t)\, u_\sigma(t)\|_{L^p_t L^p_{x'}}$

$\sigma \simeq h$ is hard b/c of interactions.

$x_1 = t$, $x_2 = t$, $t = (t_1, t_2)$, $x = (t_1, t_2, \bar{x})$.

$$\|u(t_1)\, u(t_2)\|_{L^p_{t_1} L^p_{t_2} L^p_{\bar{x}}}$$

Example: