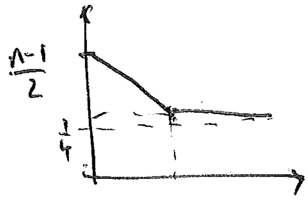
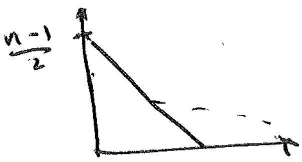
MTAC - Lecture 4. 10/07/2015.

Last time: $\|u\|_{L^p(H)} \lesssim h^{-\delta(n,p)} \|u\|_{L^2(M)}$.



$H = \{x : x_1 = 0\}$.

Case 1: localised near $\partial_{\xi_1} p \neq 0$.



Case 2: localised near $\partial_{\xi_1} p(x,\xi) = 0$.

Estimate $\|W(t)u\|_{L^p_t L^p_x}$ by

$\|W(t)W^*(s)\|_{L^2\to L^\infty}$, $\|W(t)W^*(s)\|_{L^2\to L^2}$.

$$W(t)W^*(s)u = \int w(t,s,\bar x,\bar z)\, u(\bar z)\, d\bar z.$$

$$w(t,s,\bar x,\bar z) = \frac{1}{h^{n-1}} \int e^{\frac{i}{h}(\varphi(t,\bar x,\xi') - \varphi(s,\bar z,\xi'))}\, b(t,s,\bar x,\bar z,\xi')\, d\xi'.$$

$$\varphi_t + a(t,x',\nabla_{x'}\varphi) = 0.$$

Model: $\varphi(t,x,\xi') = \langle \bar x, \xi' \rangle + |\xi'|^2$.

$$\nabla_{\xi'}(\varphi(t,\bar x,\xi') - \varphi(s,\bar z,\xi')) = \nabla_{\xi'}\big(\langle \bar x - \bar z, \xi + O(|t|)\rangle + (t-s)(a(t,x,\xi') + O(|t|-|s|))\big).$$

Model: $= \nabla_{\xi'}(\langle \bar x - \bar z, \bar\xi\rangle + (t-s)|\xi'|^2)$. $\partial^2_{\xi\xi'} a$ pos. def.

Model: $\bar x - \bar z = 2(t-s)\bar\xi$ $\quad 0 = \xi, \Rightarrow \xi = \frac{\bar x - \bar z}{(t-s)}$.

Hessian $2(t-s)\mathrm{Id}$.

General Case: No critical point ~~unless~~ unless

$$|\bar{x}-\bar{z}| < k|t-s|.$$

Hessian $(t-s)\cdot(\text{pos. def})$.

$$(*)\ W(t,s,\bar{x},\bar{z}) = h^{-\frac{(n-1)}{2}} (h+|t-s|)^{-\frac{n-1}{2}} e^{\frac{i}{h}\varphi(t,s,\bar{x},\bar{z})} B(t,s,\bar{x},\bar{z}).$$

B supported $|\bar{x}-\bar{z}| < k|t-s|$; $|D^{\gamma}_{\bar{x},\bar{z}} B| \lesssim |t-s|^{-|\gamma|}$.

Model: $\varphi(t,s,\bar{x},\bar{z}) = -(\bar{x}-\bar{z})^2/2(t-s)$

So, get from $(*)$ here:

$$\|W(t)W^*(s)\|_{L^1\to L^\infty} \lesssim h^{-\frac{n-1}{2}} (h+|t-s|)^{-\frac{n-1}{2}}.$$

For $L^2 \to L^2$:

Model: $\frac{1}{2(t-s)}\left((\bar{x}-\bar{z})^2 - (\bar{x}-\bar{y})^2\right)$.

$\frac{1}{(t-s)}\bar{x}(\bar{y}-\bar{z})$.

$\nabla_{\bar{x}} = (\bar{y}-\bar{z})/(t-s)$.

IBP $\frac{h(\bar{y}-\bar{z})}{t-s}$, $|t-s|^{-1} \sim \frac{h}{(\bar{y}-\bar{z})}$.

$$|\tilde{W}| \leq h^{-(n-1)}(h+|t-s|)^{-(n-1)}\left(1+\frac{|\bar{y}-\bar{z}|}{h}\right)^{-N}\int \underline{\qquad}.$$

$$\leq h^{-(n-1)}(h+|t-s|)^{-1}\left(1+\frac{|\bar{y}-\bar{z}|}{h}\right)^{-N} |\bar{x}-\bar{z}| \ (\leq k|t-s|).$$

$$\|W(t)W^*(s)\cdot\|_{L^2}^2 \lesssim h^{-(n-1)}(h+|t-s|)^{-1}\, h^{n-2}.$$
$$\lesssim h^{-1}(h+|t-s|)^{-1}$$

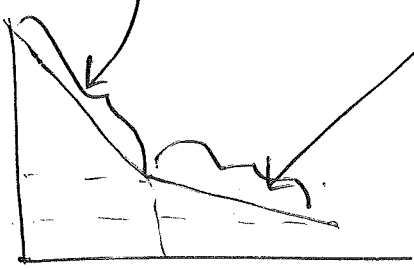
So, $\|W(t)W^*(s)\|_{L^2\to L^2} \lesssim h^{-\frac{1}{2}}(h+|t-s|)^{-\frac{1}{2}}$.

Interpolate: $\|W(t)W^*(s)\|_{L^{p'}\to L^p} \lesssim h^{-2/p}(h+|t-s|)^{-2/p}$

$L^{p/2}$ norm of $h^{-2/p}(h+\tau)^{-2/p}$; integrand

$(h+\tau)^{-(\text{big than } 1)}$
big cont $|\tau|<h$.

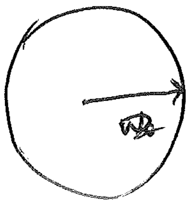
$(h+\tau)^{-(\text{less than } 1)}$.
big cont.
$|\tau|\sim 1$.



Examples: $|\xi|^2-1=0$.

Fourier side: $\mathcal{F}_h(u) = \frac{1}{(2\pi h)^{n/2}}\int e^{-\frac{i}{h}x\cdot\xi}\, u(x)\, dx$.

normalize here so $\|\mathcal{F}_h(u)\|_2 = \|u\|_{L^2}$.



w_0 correspond to ξ_1 direction.

$$\chi_\alpha(r,w) = \begin{cases} 1 & |r-1|<h,\ |w-w_0|<h^\alpha. \\ 0 & \text{otherwise}. \end{cases}$$

$$f_\alpha = h^{-\frac{1}{2}}\, h^{-\alpha\frac{(n-1)}{2}}\, \chi_\alpha$$
$$u_\alpha = \varphi(x)\, \mathcal{F}_h^{-1}(f_\alpha).$$

$\|u_\alpha\|_{L^2} = 1.$

$\|(h^2\Delta - 1)u_\alpha\|_{L^2} \lesssim h.$

$$u = \frac{\rho(x)\, e^{\frac{ix_1}{h}}}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}(x_1(\xi_1-1) + x'\cdot\xi')} \chi_\alpha(v,w)\, dv\, dw.$$

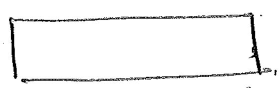
$|\xi_1 - 1| < h^{2\alpha}$ $\quad |\xi'| < h^{\alpha}$, $\quad |x'| < h^{1-\alpha}$.

So if $|x_1| < h^{1-2\alpha}$, no oscillation, treat $e^{\frac{i}{h}(x_1(\xi_1-1)+x'\xi')}$ as 1.

On $|x_1| < h^{1-2\alpha}$, $|x'| < h^{1-\alpha}$,

$$|u|_{L^\infty} > h^{-\frac12}\, h^{-(\frac{n-1}{2})\alpha}\, h^{-n/2} \cdot h \cdot h^{(n-1)\alpha} = h^{-(\frac{n-1}{2})}\, h^{(\frac{n-1}{2})\alpha} = h^{-\frac{n-1}{2}+\frac{\alpha(n-1)}{2}}$$

p large sharp here.



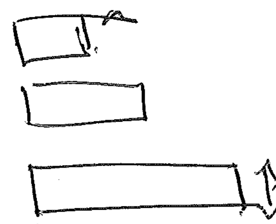
$\uparrow\downarrow h^{1-\alpha}$.

$h^{1-2\alpha}$.

$\rightarrow$ O $h^{-\frac{n-1}{2}}(h^{n-1})^{\frac1p}$.

$$\|u\|_{L^p(M)} > h^{-\frac{(n-1)}{2}+\frac{\alpha(n-1)}{2}} \cdot (h^{1-2\alpha})^{\frac1p}(h^{1-\alpha})^{\frac{n-2}{p}}$$

$$> h^{-\frac{(n-1)}{2}+\frac{n-1}{p}+\alpha(\frac{n-1}{2}-\frac2p-\frac{n-2}{p})}.$$



$h^{-\frac{n-1}{4}+\frac{n-2}{2p}}$.

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