MFAC — Lecture 3 04/07/2015.

Sps u localised, $\chi_0(x,\xi)$ cptly spted. s.t.

$$u = \chi(x, hD)u + O(h^\infty)$$

and $\|p(x,hD)u\|_{L^2(M)} \lesssim h\|u\|_{L^2(M)}$.

$p(x,\xi)$ Laplace like: ($\text{Lap} = |\xi|^2 - 1$).

- If $p(x_0,\xi_0)=0$, $\nabla_\xi p(x_0,\xi_0)\neq 0$.
- $\{\xi : p(x_0,\xi)=0\}$ has pos. def. 2nd f.f.

<u>Question</u>: How big is u on a hypersurface:

$$\|u\|_{L^p(H)} \lesssim G(n,p,h)\|u\|_{L^2(M)}.$$

Aim: find G sharp.

<u>Semi-Classical Sobolev</u>

$$\|u\|_{L^p(H)} \lesssim h^{-n/2 + \frac{n-1}{p}}\|u\|_{L^2(M)}.$$

$$u = \chi_0(x,hD) + O(h^\infty)$$

$L^2 \to L^p$ mapping norm of $\chi_0(x,hD)$.

$$\chi(x,hD)u = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\langle x-y,\xi\rangle}\chi(x,\xi)u(y)\,d\xi\,dy.$$

phase stat only when $x=y$

IBP:

$$\chi(x, hD)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} \left(1+\frac{|x-y|}{h}\right)^{-N} \hat{\chi} u \, dy \, d\xi .$$

$$\|\chi(x,hD)u\|_{L^\infty} \lesssim h^{-n} h^{n/2} \|u\|_{L^2(M)} \lesssim h^{-n/2} \|u\|_{L^2(M)} .$$

$$\|\chi(x,hD)u\|_{L^2} \lesssim h^{-n} \left\| \left(1+\frac{|x-y|}{h}\right)^{-N} \right\|^{\frac{1}{2}}_{L^1_y} \left\| \left(1+\frac{|x-y|}{h}\right)^{-N} \right\|^{\frac{1}{2}}_{L^1_x} \|u\|_{L^2}$$

$$\lesssim h^{-n} h^{n/2} \cdot h^{\frac{n-1}{2}} \|u\|_{L^2} \lesssim h^{-\frac{1}{2}} \|u\|_{L^2} .$$

Then interpolate.

__Theorem (T)__ $H = \{x : x_i = 0\}$, u localised $O_{L^2}(h)$ QM of $p(x, hD)$ (LL operator).

$$\|u\|_{L^p(H)} \lesssim h^{-\delta(n,p)} \|u\|_{L^2(M)} .$$

$$\delta(n,p) = \begin{cases} \frac{n-1}{2} - \frac{n-1}{p} & \frac{2n}{n-1} \le p \le \infty . \\ \frac{n-1}{4} - \frac{n-2}{2p} & 2 \le p \le \frac{2n}{n-1} . \end{cases}$$

__Note:__ L^2 estimate on all of M is trivial. But for hypersurface, it is __not__: $\|u\|_{L^2(H)} \lesssim h^{-\frac{1}{4}} \|u\|_{L^2(M)}$.

$\nabla_\xi p(x_0, \xi_0) \neq 0$. Some ξ_i s.t. $\partial_{\xi_i} p(x_0, \xi_0) \neq 0$.

__Case 1:__ $\xi_i = \xi_1$; $\partial_{\xi_1} p(x_0, \xi_0) \neq 0$.

near (x_0, ξ_0), $p(x,\xi) = e(x,\xi)(\xi_1 - a(x,\xi))$.

$|e(x,\xi)| > c > 0 \Rightarrow e(x, hD)$ invertible.

$$\|(hD_{x_1} - a(x, hD_{x'}) u\| _{L^2(M)} \lesssim h \|u\|_{L^2(M)}$$

Case 2: $\xi_i \neq \xi_1$; $\partial_{\xi_1} p(x_0, \xi_0) = 0$, $\partial_{\xi_2} p(x_0, \xi_0) \neq 0$.

$$p(x,\xi) = e(x,\xi)(\xi_2 - a(x, \xi')), \quad e(x, hD) \text{ inv.}$$

$$\|(hD_{x_2} - a(x, hD_{x'}) u\|_{L^2(M)} \lesssim h \|u\|_{L^2(M)}.$$

Case 1: $x_1 = t$.

$$u(t,x') = \mathcal{U}(t,0) u(x_0, x') + \frac{1}{h}\int_0^t \mathcal{U}(t-s,s) E[u]\, ds.$$

L^2 away in $0 < t < 1$.

$$\|u\|_{L^2(M)} \lesssim \|u\|_{L^2(M)} + \frac{1}{h}\|E[u]\|_{L^2(M)} \lesssim \|u\|_{L^2(M)}.$$

L^∞ estimate: $\|u\|_{L^\infty} \lesssim h^{-\frac{n-1}{2}} \|u\|_{L^2(M)}$

$$\|u\|_{L^p} \lesssim h^{-\frac{n-1}{2}+\frac{n-1}{p}} \|u\|_{L^2(M)}.$$

Case 2: $x_2 = t$.

$$\|(hD_t - a(t, x', hD_{x'}) u\|_{L^2(M)} \lesssim h\|u\|_{L^2(M)}.$$

$$(hD_t - a(t,x', hD_{x'}) u = h f(t,x').$$

$x = (x, t, x')$

$$u(x,t,\bar{x}) = \mathcal{U}(t,0) u(x, 0, \bar{x}) + \int_0^t \mathcal{U}(t-s,s) f(x,s,\bar{x})\, ds.$$

$x_1 = 0$ $\quad W(t,s) = \mathcal{U}(t,s)\big|_{x_1=0}$. Want $\|W(t)g\|_{L^p_t L^p_{\bar{x}}}$

$$W(t)u = \frac{1}{(2\pi h)^{n-1}} \int e^{\frac{i}{h}(\varphi(t,x',\xi') - \langle y', \xi' \rangle)} b(t,x',\xi') u(y) \, d\xi' \, dy.$$

Model case: $\xi_2 - |\xi'|^2$.

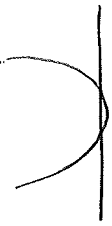
$$\varphi_t + a(t,x,\nabla_x \varphi) = 0.$$

$$\varphi(0,x',\xi') = \langle x', \xi' \rangle$$

$$\varphi(t,x',\xi') = \langle x', \xi' \rangle + t\, a(t,x,\xi') + O(t^2)$$

$\dot{x}(t) = 2\xi'$

$\dot{\xi}' = 0.$



Model: $\varphi(t,x',\xi') = \langle x', \xi' \rangle + |\xi|^2.$

Need: $W(t)W^*(s)$. and $L^1 \to L^\infty$, $L^2 \to L^2$.

$$\cancel{W(t)W^*(s)} \; W(t,s,x,z) = \frac{1}{(2\pi h)^{2(n-1)}} \int e^{\frac{i}{h}(\varphi(t,x,\xi') - \langle y', \xi' \rangle + \langle y', \eta' \rangle - \varphi(s,z,\eta'))}. \; B \, dy' \, d\eta' \, d\xi'$$

where $W(t)W^*(s)g = \int W(t,s,x,z)\, g(z)\, dz.$