Semiclassical eigenfunction estimates — MTAC. Lecture 2.

08/07/2015.

QM (Quasi-modes).

Def. u is an $O_{L^2}(h^\alpha)$ QM. of $p(x,hD)$ if

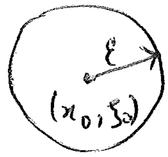
$$\|p(x,hD)u\|_{L^2} \lesssim h^\alpha \|u\|_{L^2}.$$

Usually is $O_{L^2}(h)$ – i.e. $\alpha = 1$.

Sps $\chi(x,\xi)$ cptly sptd in both x and ξ.

u is an $O_{L^2}(h)$ QM of $p(x,hD)$.

$$p(x,hD)\underbrace{\chi(x,hD)u}_{\text{localised.}} = ?\ .$$



Semi-Classical analysis says that.

$$? = \chi(x,hD)p(x,hD)u + \underbrace{hr(x,hD)u}_{O_{L^2}(h)}.$$

$$\Rightarrow \|p(x,hD)\chi(x,hD)u\|_{L^2} \lesssim h\|u\|_{L^2}.$$

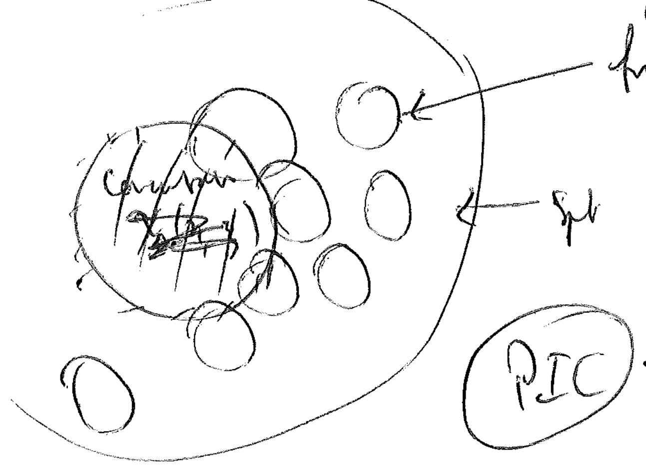
Assume $\exists \chi_0(x,\xi)$ (cpct spt, but large).

$$\chi_0(x,hD)u = u + O_{L^2}(h^\infty).$$

$$(h^2\Delta - 1)u = O_{L^2}(h).$$

(Reasonable b/c large spt. means away from concentration no information).

Cover by further small balls of around ε-length



$\leftarrow$ supp $\chi_0(x, \xi_0)$.

PIC.

$$u_\lambda = \sum_{\lambda_j \in [\lambda, \lambda+1]} c_j u_j . \qquad \Delta u_j = \lambda_j^2 , \quad \text{let } h = \lambda^{-1}.$$

$\lambda_j \in [\lambda, \lambda+1]$. $\leftarrow$ spectral window.

$$(h^2\Delta - 1)\, u_\lambda = \sum_{\lambda_j \in [\lambda,\lambda+1]} (h^2 \lambda_j^2 u_j - u_j)\cdot c_j .$$

$$\| (h^2\Delta - 1) u_\lambda \| \leq \left(\sum c_j^2 3h^2 \right)^{\frac{1}{2}} \lesssim 3h .$$

Localisation by small balls in PIC, only important to consider $p(x_0, \xi_0) = 0$ because:

Suppose $p(x_0, \xi_0) \neq 0$, pick χ supported on (x_0, ξ_0) and $|p(x,\xi)| > c > 0$.
(by continuity, this estimate is valid).

$$\| p(x, hD)\chi(x, hD) u \|_{L^2} \lesssim h \|u\|_{L^2} .$$

and $p(x, hD)$ invertible because of $\longrightarrow$ in this part of phase space. $\Rightarrow$

$$\| \chi(x, hD) u \|_{L^2} \lesssim h \|u\|_{L^2} .$$

(If u was perfect eigenfunction, upto order h^∞.
But u is only QM so order h is best we can do)

Semi-Classical Evolution op.

Assume: $\begin{cases} (hD_t + p(t,x,hD_x))u(t) = 0 \\ u(0) = Id. \end{cases}$

then, $\mathcal{U}(t)u = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(\varphi(t,x,\xi) - \langle y,\xi\rangle)} b(t,x,\xi).\, u(y)\, dy\, d\xi.$

$$\begin{cases} \varphi_t + p(t,x,\nabla_x\varphi) = 0 \\ \varphi(0,x,\xi) = \langle x,\xi\rangle. \\ b(0,x,\xi) = 1. \end{cases}$$

N.B. y only appears in the phase, not in the parametrix b.

$\mathcal{U}(0)u = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y)\,dy\,d\xi = u$; ie $\mathcal{U}(0) = Id.$

~~$\mathcal{U}(0) = Id.$~~

$$hD_t\,\mathcal{U}(t) = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(\varphi(t,x,\xi) - \langle y,\xi\rangle)}\left[\varphi_t\, b(t,x,\xi) + hD_t b\right] d\xi\, dy.$$

$$p(t,x,hD)\,\mathcal{U}(t) = \frac{1}{(2\pi h)^{2n}}\int e^{\frac{i}{h}\psi(x,z,y,\xi,\eta)} p(x,\eta)\, b(t,z,\xi).\, d\eta\, dz\, d\xi\, dy.$$

$$\psi(x,z,y,\xi,\eta) = \langle x-z,\eta\rangle + \varphi(t,z,\xi) - \langle y,\xi\rangle.$$

Compute ~~Integrate in~~ integral in z and η to that ~~remaining~~ expression is same as $hD_t\,\mathcal{U}$~~(t)~~ expression.

critical pts of ψ are non-deg:

integral$(z,\eta) = (2\pi h)^n e^{i\psi_c}(p.b|_c + hr)$.

Critical pt means:

$\partial_{z_i}\psi = 0 \Rightarrow \quad \eta_i = \partial_{z_i}\varphi$.

$\partial_{\eta_i}\psi = 0 \Rightarrow \quad x_i - z_i = 0 \Rightarrow \quad x_i = z_i$.

$\partial^2_{\eta_i z_i} = 1 \quad \partial^2_{\eta_i z_j} = 0, \ i \neq j$.

So, $p(t,x,hD) = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(\varphi(t,x,\xi) - \langle y,\xi\rangle)}$.

$$\Big[\underbrace{p(x, \nabla_x\varphi)\, b(t,x,\xi)}_{\text{Important term}} + \underbrace{hr(x,\xi)}_{\text{order } h \text{ better.}}\Big]\, dy d\xi.$$

So, h^0 term:

$$\left(hD_t + p(t,x,hD)\right)u(t) = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\left(\varphi(t,x,\xi) - \langle y,\xi\rangle\right)}\cdot$$

$$\left[\varphi_t + p(t,x,\nabla_x\varphi)\right]\cdot b(t,x,\xi)\, d\xi dy$$

pick φ so that ~~this~~ this $=0$.

Let $b(t,x,\xi) = \sum_{j=0}^\infty h^j b_j(t,x,\xi)$. if $b^N = \sum_{j=0}^N h^j b_j$.
$\rightarrow |b - b^N| \leq h^N$.

$b_0(0,x,\xi) = 1$.
$b_j(0,x,\xi) = 0 \quad i \neq 1$.

Order h term: $\quad h\big(D_t b_0 + \underbrace{r_1(x,\nabla_x\varphi)}_{\text{Stationary phase.}}\big)$.

Can have derivations of b_0

Rmk. $\lesssim_{N,M,T} h^N$. So, stop at N depends on max time T.

So, solve away but <u>need</u> to stop at some N, because as $N \to \infty$, the time for which the expression is valid $\to 0$.

<u>Remark 2</u>: $N = c \dim(M)$. typically is good enough for most applications. $c = 10$, say.

Eigenfunction. $u = U(t) u$.

Qu. $u = U(t)u + \frac{1}{h}\int_0^t U(t-s, s) E[u]$.

Always take TT^* trick:

$\langle v, Tu \rangle = \langle T^* v, u \rangle$.

$\|TT^*\|_{L^{p'} \to L^p} \leq \|T\|^2_{L^2 \to L^p}$.

$\|T^* u\|^2_{L^2} = \langle T^* u, T^* u \rangle = \langle TT^* u, u \rangle$. $\forall u$.

$\Rightarrow \|T^*\|^2_{L^{p'} \to L^2} \lesssim \|TT^*\|_{L^{p'} \to L^p}$.

Needed for Strichartz estimates.

Strichartz Estimates:

Need $\|W(t)u\|_{L_t^{p^*} L_x^p}$ estimates.

Keel–Tao: $U(t): H \to L^p$. H – Hilbert Space.

$U(t)$ unitary: $H \to L^2$. (think of $H = L^2$).

depend.

$\|u(t)u^*(s)\|_{L^1 \to L^\infty} \lesssim |t-s|^{-\gamma}$.

① dispersion bound.

$\|u(t)u^*(s)\|_{L^1 \to L^\infty} \lesssim |t-s|^{-\gamma}$.

② $\|u(t)u^*(s)\|_{L^2 \to L^2} \lesssim 1$.

③ Interpolate: $\|u(t)u^*(s)\|_{L^{p'} \to L^p} \lesssim |t-s|^{-\gamma/p}$.

④ Hardy-Littlewood-Sobolev to resolve $|t-s|^{-\gamma/p}$.

In semi-classical setting:

① truncation estimate $h^{-\mu_\infty}(h + |t-s|)^{-\gamma_\infty}$.

(So no singularity at $t=s$ as in KT).

② Not unitary. get an $L^2 \to L^2$ estimate.

of the form $h^{-\mu_2}(h + |t-s|)^{-\gamma_2}$.