A semi classical approach to eigenfunctions.

08/07/2015.

MTdc.

Lecture 1.

Why eigenfunctions?

Quantum.

$$\frac{\hbar}{i}\partial_t \psi(t,x) = \hat{H}\psi(t,x).$$

$\|\psi\|_{L^2}$ interpreted probabilistically.

physics of the small.

Classical.

$$\dot{x}(t) = \nabla_\xi H(x,\xi).$$
$$\dot{\xi}(t) = -\nabla_x H(x,\xi).$$

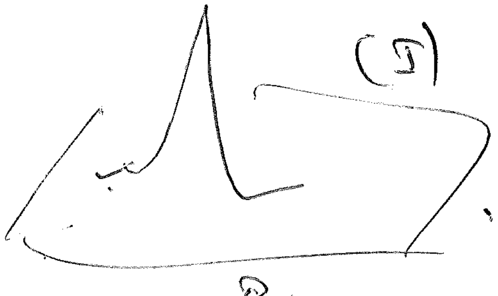
physics of the large.

Should "see" classical come from Quantum as we go $\longrightarrow$.

$$\psi(t,x) = e^{\frac{i}{\hbar}t\lambda^2}u(x), \quad \hat{H}u = \lambda^2 u.$$

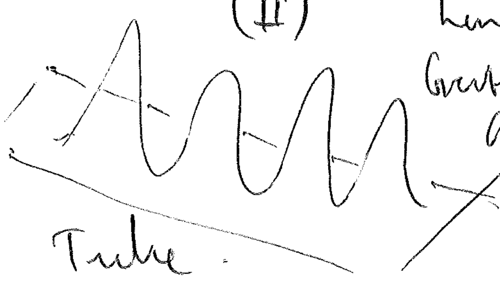
- λ^2 interpreted as energy.

Two Concentrations.



(I) High L^∞. Sharper change in L^p, $p<\infty$.

Point



(II) Lower L^∞. Greater L^p change.

Tube.

For (II), tubes, cross sections allow ~~better~~ study of L. $\Rightarrow$ Hyper surface estimates.

Bilinear est.: $\hat{H}u = \lambda^2 u$, $\hat{H}v = \mu^2 v$.

What is $\|u \cdot v\|_{L^2}$?

Semi classical picture: $\xi_i \to hD_{x_i}$, $|\xi|^2 \to h^2\Delta$.

$h = \lambda^{-1} \Rightarrow$
$$\begin{cases} (h^2\Delta - 1)u = 0 \\ \Delta u = h^{-2}u = \lambda^2 u. \end{cases}$$

Can work with symbol: $p(x, hD) = \frac{1}{(2\pi h)^n}\int e^{i/h\langle x-y,\xi\rangle} p(x,\xi)u(y)\,d\xi\,dy$

Quantum-Classical correspondence: $\tilde{p}(x,\xi)$ Hamil. Flow:
$$\begin{cases} \dot{x}(t) = \partial_\xi \tilde{p}(x,\xi) \\ \dot{\xi}(t) = -\partial_x \tilde{p}(x,\xi). \end{cases}$$

Expect to see classical flow in
$$\tilde{p}(x, hD)u = u.$$

Better reparametrisation. $p(x, hD)u = 0$, $p(x,\xi) = \tilde{p}(x,\xi) - 1$.

Why semi-classical?

- Clean way to think about localisation - adding up contributions from diff. bits of phase space.

- Encodes energy into quantisation (rather than symbol).
  $\rightarrow$ useful ~~Hobben~~ bilinear estimates are needed when ~~two~~ coming from two diff. scales.

- Allows use of FT.

Definition: $e^{\frac{it}{h}p(x,hD)}$ quantisation of classical flow.

Sps $p(x,hD)u = 0$

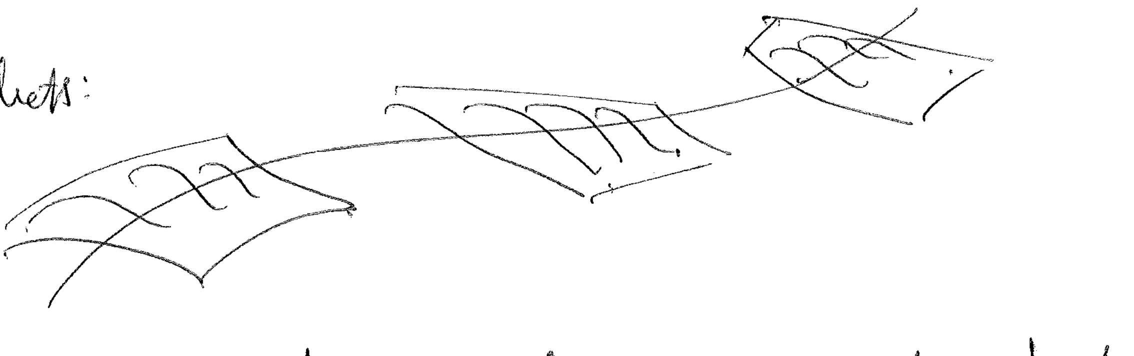
$$(hD_t + p(x,hD))u = 0.$$

$$\Rightarrow \quad u = e^{\frac{it}{h}p(x,hD)}u \quad \forall t > 0.$$

then

$$u = \int \chi\left(\frac{t}{T}\right) e^{\frac{it}{h}p(x,hD)} u \, dt.$$

Wave packets:



Think of eigenfns being made of wave packets tracking classical flow

- localized in frequency and space.
- Concentration related to time packets spread more.
- ~~Heur~~ long time: dispersion wavelets heuristic.

<u>Quasimodes</u>: cut offs of eigenfunctions are <u>not</u> eigenfunctions. To lose localisation, need to study quasimodes u

$$p(x,hD)u = O_{L^2}(h^{\alpha}).$$

(normally $\alpha = 1$).

(Exact eigenfunctions, or even estimates, are difficult to come by!)

<u>Quasi-inv. for quasimodes.</u>

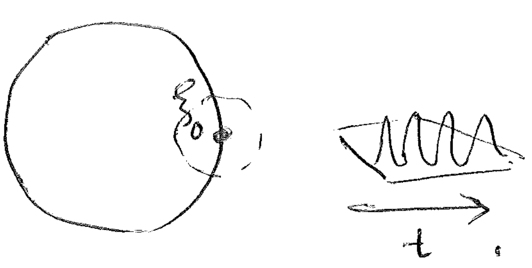
Sps $p(x,hD)u = E[u] \Rightarrow (hD_t - p(x,hD))u = E[u]$.

Duhamel $\Rightarrow$ $u = e^{\frac{it}{h}p(x,hD)}u + \frac{i}{h}\int_0^t e^{\frac{i(t-s)}{h}p(x,hD)} E[u]\, ds.$

Error term limits how long we can average in time.

$O_{L^2}(h)$ quasimodes, run out time is $T = \varepsilon$.

Consider. $(h^2\Delta_{\mathbb{R}^2} - 1)u = O_{L^2}(h)$. $\Rightarrow$ $\mathcal{F}_h[u]$ lives near $|\xi|^2 = 1$.
(semi-classical Fourier transf.).



Localise to small region near ξ_0.
Say ξ_0 principal direction.

<u>Using principal direction:</u> variable freezing trick.

$$Tu = \int e^{i\lambda\varphi(x,y)} a(x,y)\, u(y)\, dy.$$

studied in Harm. Anal.

(I) $p(x_0,\xi_0) = 0$, and $\nabla_\xi p(x_0,\xi_0) \neq 0$.

(II) $\{\xi \mid p(x_0,\xi) = 0\}$ is <u>pos. def.</u> (pos. curvature).

(I) $\Rightarrow \exists\, \xi_i,\ |\partial_{\xi_i}(p(x,\xi))| > c > 0$. (near (x_0,ξ_0)).

So implicit func thm:

$$p(x,\xi) = e(x,\xi)(\xi_1 - a(x,\xi'))$$ where $\xi_i = \xi_1$ w.l.o.g.

where $|e(x,\xi)| > c > 0$.

(Factorised out one direction ~ the principal direction.)

So. $e(x,hD)(hD_{x_i} - a(x,hD_{x'}))u = o_{L^2}(h)$.

$\Rightarrow$ $(hD_{x_i} - a(x,hD_{x'}))u = o_{L^2}(h)$.

Setting $x_i = t$, evolution in t, ~~but does not depend on variable decision~~

So factored at the principal direction of oscillation. and <u>freeze this variable</u>. So,

$$\text{} = u_n(x,0) + \frac{i}{h}\underbrace{\int_0^t \underline{\qquad\qquad}}_{\text{Duhamel term}}$$

<u>Gameplan:</u> Hypersurface est.

Suppose $p(x,hD)u = O_{L^2}(h)$. Want L^p bounds on restriction of u to $H = \{x : x_1 = 0\}$.

(1) Factorise $p(x,hD)$ and set $x_1 = t$. matters $i = 1$ and $i \neq 1$.

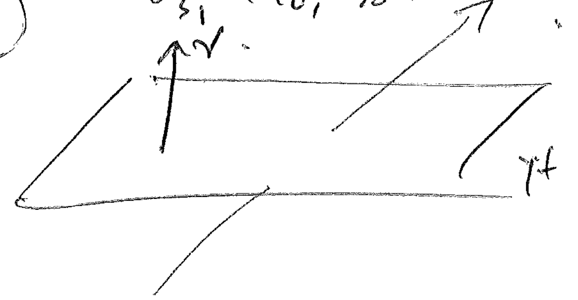
(2) Duhamel for [illegible] in terms of $U_h(t,s)$.

(3) Slides $\longrightarrow$ Strichartz $L^p_t L^p_{x'}$

(4) Bilinear est.

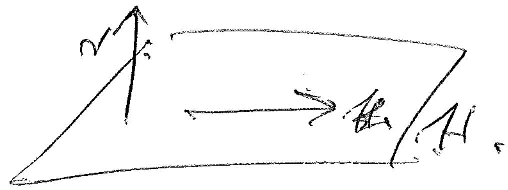
(1)

$\partial_{\xi_1} p(x_0, \xi_0) \neq 0$:



no concentration.

$\partial_{\xi_1} p(x_0, \xi_0) = 0$.



concentration.

② Parametrise sol^n for $U_h(t,s)$.

$$U_h(t,0)u = \frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}\phi(t,x',\xi)-\langle y',\xi'\rangle} \underbrace{b(\quad)}_{\text{param.}}\cdots$$

③. Need $L^p_t L^p_x$ norms for $W_h(t,s)$.

$$\begin{cases} \| W(t,0)W^*(s,0)\|_{L^1\to L^\infty} \lesssim h^{-\mu_0}(h+|t-s|)^{-r_\infty} \\ \| W(t,0)W^*(s,0)\|_{L^2\to L^2} \lesssim h^{-\mu_2}(h+|t-s|)^{-r_2} \end{cases}$$

Two fundamental estimates.

diff. scales.

④ $P(x,hD)u = O_{L^2}(h)$, $P(x,D)v = O_{L^2}(h)$.

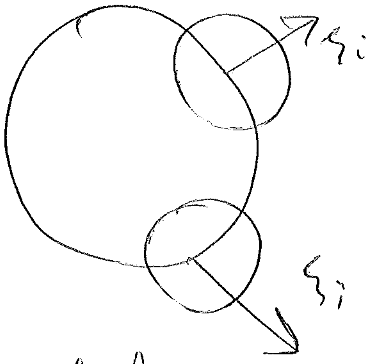
Want L^p bounds on $u\cdot v$.

Factorisation same way $\begin{cases} \text{— prop. in the same direction.} \\ \text{otherwise} \end{cases}$

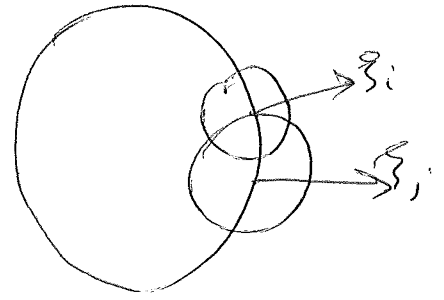
Dilation $U_h(t,s)$ $U_\sigma(t,s)$.

product $u\cdot v \rightsquigarrow$ restriction of $u(x)v(y)$ to submfld $x=y$.

Look at Strichartz.

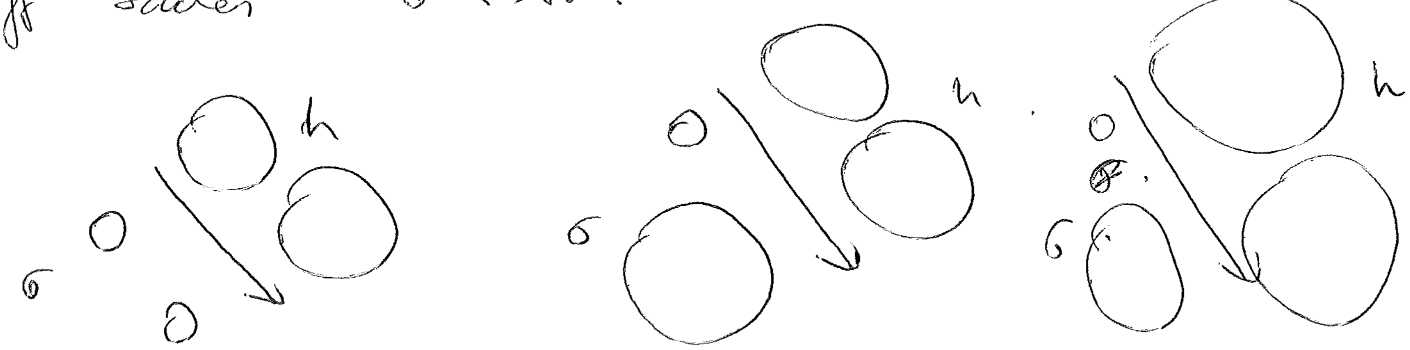


diff direcn.



Same direction.
(dont need to be exact same,
but feature has same exp.)

Diff scales $\sigma < h$.



No dispersion.
(nothing has happend)

dispn for σ.

dispn for both.

t

$\boxed{\text{regions to analyse.}}$