Ingo Witt (Q & A) 08/07/2015.

Stationary phase. $I_h = I_h(a,\varphi) = \int_{\mathbb{R}^d} e^{i\varphi(x)/h} a(x)\, dx$.

$\varphi \in C^\infty(\mathbb{R}^d)$ real valued, $a \in C_c^\infty(\mathbb{R}^d)$.

If $\exists!$ critical pt x^0 of φ in spt a, it's nondegenerate

(I). $\exists$ diff ops $A_{2k}(x,D)$ s.t. $\forall K$

$$\left| I_h - \left(\sum_{k<K} [A_{2k}(x,D)a](x^0)\, h^{k+d/2}\right) e^{i\varphi(x^0)/h} \right|$$

$$\lesssim h^{K+d/2} \sum_{|\alpha| = 2K+d+1} \sup |\partial^\alpha a|$$

(II). $A_0 = (2\pi)^{d/2} |\det \operatorname{Hess}\varphi(x^0)|^{-\frac{1}{2}} e^{i\pi/4 \operatorname{sgn} \operatorname{Hess}\varphi(x^0)}$.

Pf ⓪ No critical pts $\Rightarrow I_h = O(h^\infty)$.

① Morse lemma: $\varphi: \mathbb{R}^d \to \mathbb{R}$ $\in C^\infty$ non-deg. critical pt x^0. $\Rightarrow \exists$ local coord centred at x^0,

$$\varphi(x) = \varphi(x^0) + \tfrac{1}{2}\left(x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_d^2\right).$$

② $\varphi(x) = \frac{1}{2}\langle Qx, x\rangle$ Q non sing, symm. real.

- $F\left(e^{\frac{i}{2}\langle Qx,x\rangle}\right) = \dfrac{(2\pi)^{d/2}}{|\det Q|^{\frac{1}{2}}} e^{i\pi/4 \operatorname{sgn} Q} e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}$.

- $\langle Fu, v\rangle = \langle u, Fv\rangle = \underbrace{\qquad\qquad}_{\text{taylor expand}}$

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Simon Marshall (expanding Pohl's lectures).

$\mathcal{D}(X)$ commutative.

A, $\mathfrak{a} = \mathrm{Lie}(A)$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}/W$.

$D \in \mathcal{D}(X) \cong \mathrm{Poly}(\mathfrak{a}^*)^W$. Thm. Harish-Chandra.

$D \rightsquigarrow p$.

$\psi \in C^\infty(X) \not\Rightarrow D\psi = p(\lambda)\psi$.

$G = SL(3,\mathbb{R})$, $o \in X = G/K = SL(3,\mathbb{R})/SO(3)$.

$X = \{ g \in M_{3\times 3}(\mathbb{R}) : {}^t g = g, g > 0, \det g = 1 \}$.

$I \in X$.

$g \in SL(3,\mathbb{R})$, $g I {}^t g = g {}^t g = I$.

$A \in X$, $g \cdot A = g A {}^t g$

$\dim X = 5$, $\mathrm{rk}\, X = 2$, diagonal matrices $\subset X$ form a flat:

$A = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \subset SL(3,\mathbb{R})$ $A > 0$.

$W = N_G(A)/Z_G(A)$. $Z_G(A) = \tilde{A} = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$ +ve and -ve.

$N_G(A) = \tilde{A} \times S_3$ symm on 3 symbols.

$\Rightarrow W = S_3$.