Semiclassical Analysis. – Ingo Witt. 07/07/2015.

What is SA? – Study properties of sol^{n}
$u = \{u_h\}_{0<h\leq h_0}$ of $P_h u_h = O(h^N)$, $N \in \mathbb{Z}$.
in the limit as $h \to 0$ with microlocal methods.

$P_h = p(x, hD; h)$. $\longleftarrow$ symbol.

Example: Schrödinger eq^{n}.

$$i h \partial_t u = -h^2 \Delta u + V(x,h).$$

$$p(t,x,\tau,\xi) = \tau + |\xi|^2 + V(x,h).$$

$$\tau \rightsquigarrow hD_t = -ih\partial_t, \quad \xi_j \rightsquigarrow hD_{x_j}.$$

(I) Basic structures:

FT: $(F_h u)(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi/h} u(x)\, dx.$

1) Plancherel: $F_h : L^2(\mathbb{R}^d, dx) \xrightarrow{\cong} L^2(\mathbb{R}^d, (2\pi h)^{-d} d\xi)$.

2) $(F_h^{-1} v)(x) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{ix\xi/h} v(\xi)\, d\xi.$

3) $F_h(hD_{x_j} u)(\xi) = \xi_j F_h u.$

$A = a(x, hD; h)$ – diff. op, $a(x,\xi;h) = \sum_{|\alpha|\leq m} a_\alpha(x;h)\xi^\alpha$.

$\Rightarrow$ ~~used~~ $\underbrace{A = \mathcal{F}_h^{-1} a(x, \xi; h) \mathcal{F}_h.}_{\text{Std quantisation.}}$

Quantisation: $\left(a^w(x, hD; h)u\right)(x)$.

$$= (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\xi/h} \int a\left(\frac{x+y}{2}, \xi; h\right) u(y)\, dy\, d\xi.$$

$\boxed{\text{Weyl quantisation}}$

Admissible weights. $M \in C^\infty(\mathbb{R}^{2d})$ $0 < M(x, \xi) \lesssim \langle x, \xi\rangle^N$

Japanese bracket. $= (1 + |x|^2 + |\xi|^2)^{\frac{1}{2}}$.

$$|\partial_x^\alpha \partial_\xi^\beta M(x, \xi)| \lesssim M(x, \xi) \quad \forall \alpha, \beta.$$

Ex. $M = \langle \xi \rangle^m$, $\langle x, \xi\rangle^m$, etc.

Symbols: $a \in S(M)$, $a = a(x, \xi, h)$, $h \in (0, h_0]$.

(1). $a(\cdot, \cdot; h) \in C^\infty(\mathbb{R}^{2n})$.

(2). $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \lesssim M(x, \xi)$ uniformly in $h \in (0, h_0]$.

$a^w = a^w(x, hD, h) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

$a^w : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$.

<u>Regular Symbols</u>: $a \in S^{reg}(M) \subset S(M)$.

$\exists \{a_j\}_{j\geq 0} \subset S(M)$, a_j – indep of h s.t.

$$\forall k \in \mathbb{N}_0 : a - \sum_{j<k} h^j a_j(x,\xi) \in h^k S(M).$$

like Taylor expansion.

$a_0(x,\xi)$ – principal symbol.
$a_1(x,\xi)$ – sub – " – .

Thm. $a \in S(1) \Rightarrow a^w : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ cont.

$\boxed{M \equiv 1}$. $\|a^w(x,hD,h)\|_{L^2 \to L^2} = \sup_{\mathbb{R}^{2d}} |a| + o(h)$.
as $h \to 0$.

<u>Composition</u> Moyal product $\#$.

$$(a\#b)^w(x,hD,h) = a^w(x,hD,h) \circ b^w(x,hD,h).$$

$a \in S(M)$, $b \in S(M') \Rightarrow a\#b \in S(MM')$.

$a \in S^{reg}(M)$, $b \in S^{reg}(M') \Rightarrow a\#b \in S^{reg}(MM')$.

$$a\#b - a_0b_0 - h\left(a_0b_1 + a_1b_0 + \frac{1}{2i}\{a_0,b_0\}\right) \in h^2 S(MM')$$

$\{a_0,b_0\} = H_{a_0}b_0 = \sum_{j=1}^d \left(\frac{\partial a_0}{\partial \xi_j}\frac{\partial b_0}{\partial x_j} - \frac{\partial a_0}{\partial x_j}\frac{\partial b_0}{\partial \xi_j}\right)$ Poisson bracket.

Hamiltonian v. field.

$$H_h(M) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : M(x, hD) \in L^2(\mathbb{R}^d) \}.$$

Sobolev ↑ weight param. (like smoothness).

$M(x, hD)$ is invertible for $h > 0$ small (by ellipticity).

Prop $a \in S(M) \Rightarrow \|a^w\|_{H_h(MM') \to H_h(M')}$. $\forall M'$ any weight function.

My Remark: Classically $H^{s+r} \xrightarrow{a} H^s$ if $a \in S^r$. Because a differentiates r times if it's in S^r symbol.

Gårding inequality: $a \in S(1)$, $a \geq 0$

$$\Rightarrow \langle a^w(x, hD, h) u, u \rangle_{L^2} \gtrsim -h \|u\|_{L^2}^2.$$

Ex: $P = -h^2 \Delta + V(x)$, $P_0(x, \xi) = |\xi|^2 + V(x)$.

$h = 1$: $\sigma^2(P)(x, \xi) = |\xi|^2$ on $T^*\mathbb{R}^d \setminus 0$.

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$.

Ellipticity: $a \in S^{reg}(M)$ is ~~the~~ elliptic if $|a_0(x,\xi)| \gtrsim M(x,\xi)$.

$$\Rightarrow \exists\, b \in S(1/M) : a \# b - 1,\ b \# a - 1 \in h^\infty S(1)$$

(Cor $a \in S^{reg}(M)$ ellip. $\Rightarrow a^w(x, hD, h)$ inv for $h > 0$ small).

(II). Application. (Weyl's law and prop. singularities).

1. Weyl's law:

$p \in S^{reg}(M)$, $P = p^w(x, hD, h)$.

- p real-valued.
- $p_0 \geq 0$
- $M = 1 + p_0$.

P_h as unbdd op. in $L^2(\mathbb{R}^d)$ is ess. s.a. with domain $\mathcal{S}(\mathbb{R}^d)$ and s.a. $H_h(M)$.

Helffer-Sjöstrand formula: $f \in C_c^\infty(\mathbb{R})$, $f(P)$.

$\exists$ almost-analytic extension $\tilde{f}$:

- $\bar{\partial}\tilde{f} = o(|\operatorname{Im} z|^\infty)$.
- $\tilde{f}|_{\mathbb{R}} = f$.

$$f(P)u = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{f}(z)(P-z)^{-1}u \, dx\,dy \quad \Big| \, z = x+iy.$$

to be safe. $\Longrightarrow$ $f(P)$ is a semi classical op.

Th^m Let $\alpha < \beta$ and sps that

$$\liminf_{|x|,|\xi|\to\infty} \operatorname{dist}(p_0(x,\xi), [\alpha,\beta]) > 0.$$

Then the spectrum of P_h is discrete in nbhd of $[\alpha,\beta]$ and

$\#\{$ eigenvalues of P_h in $[\alpha,\beta]\}$.

$$= (2\pi h)^{-d} \operatorname{vol}\{(x,\xi) : P_0(x,\xi) \in [\alpha,\beta]\} + o(1)$$

<u>Thm</u>. $f(P_h) = q^w(x, hD, h)$, $q \in S^{\text{reg}}((x,\xi)^{-N})$ $\forall N \geq 0$.

$q_0 = f(P_0)$, $\quad q_1 = P_1 f'(P_0)$

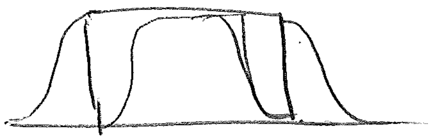
Integration.

$$(2\pi h)^d \operatorname{Tr} f(P_h) = \int_{\mathbb{R}^d} f(P_0(x,\xi))\,dx\,d\xi$$

$$+ h \int_{\mathbb{R}^{2d}} P_1(x,\xi) f'(P_0(x,\xi))\,dx\,d\xi + O(h^2).$$

<u>Pf</u> (of Weyl law).

$\#\{$ ev's in $[\alpha,\beta]\} = \operatorname{Tr} \chi_{[\alpha,\beta]}(P)$.



<u>Rmk</u> 1. If α, β are noncritical values of P_0, $o(1)$ improves to $o(h)$.

2. If the set of periodic trajectories of H_{P_0} has zero measure in $P_0^{-1}(\alpha)$ and $P_0^{-1}(\beta)$, then $o(h)$ becomes $Ah + o(h)$, with A computable in terms of P_0, P_1.

(II.2) Propagation of singularities.

$u = (u_h)$, $\|u_h\|_{H_h(M)} = O(h^{N})$ $N > 0$.

Def^n $WF_h(u) \subset T^*\mathbb{R}^d$ $(x^0, \xi^0) \notin WF_h(u)$.

if $\exists a \in S^{reg}(1)$ elliptic at (x^0,ξ^0).
(uniformly $\neq 0$ in h)

s.t. $\|a_h^w u_h\|_{H_h(M)} = O(h^\infty)$.

Ex. (I) $\varphi \in C^\infty(\mathbb{R}^d)$ real-valued, $a \in \mathcal{S}(\mathbb{R}^d)$.
$\Rightarrow WF_h\left(a(x) e^{i\varphi(x)/h}\right) = \{(x, d\varphi(x)) : x \in \operatorname{spt} a\}$.

(II) $u \in \mathcal{S}'(\mathbb{R}^d)$ is indep of $h \Rightarrow$

$WF_h(u) = (\operatorname{spt} u \times \{0\}) \sqcup WF(u) . \subset T^*\mathbb{R}^d \setminus 0$.

Prop. $WF_h(a^w u) \subset WF_h(u)$ $\forall a \in S(m)$.

Egorov's Th^m. $\{e^{itP/h}\}_{t\in\mathbb{R}}$: propagator for Schrödinger
- strongly cts unitary group in $L^2(\mathbb{R}^d)$.

$\{\varphi_t\}_{t\in\mathbb{R}}$ – flow of H_{P_0}.
Hamiltonian of P_0.

$\underline{\text{Th}^{\underline{m}}}$ $a \in S(M)$; $e^{itP/h} a^w e^{-itP/h} = a_t^w$.

- $a_t \in S(M)$
- $a_t - a_0 \circ \varphi_t \in h\, S(M)$.

$\underline{\text{Note}}$: $a \in S^{reg}(M) \longrightarrow (a_t)_0 = \varphi_t^* a_0$.

$\varkappa : (x^0, \xi^0) \to (0,0)$ germ of canonical transfm.

$\sigma = d\xi \wedge dy = d(\xi\, dx)$, $\quad \varkappa^* \sigma = \sigma$.

$[(y,\eta) = \varkappa(x,\xi) \Rightarrow d\xi \wedge dy = d\eta \wedge dy]$.

Thm. $\exists\, F : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ unitary.

s.t. $F^* a^w F = (\varkappa^* a)^w$ for all $a \in S(M)$.

microlocally near (x^0, ξ^0), $(0,0)$.

"Hyperbolic $\Rightarrow$ symbol of $\underline{\text{real prin type}}$".

$\underline{\text{Def}^{\underline{n}}}$. $p \in S^{reg}(M)$, $P = p^w$ is said to be of real principal type if P_0 is real-valued and $dP_0 \neq 0$ on $\{P_0 \neq 0\}$.

$\underline{\text{Th}^{\underline{m}}}$. P real prin. type at (x,ξ). Then $\exists\, \varkappa$ and F. as above s.t.

$F^* P F = h D_{x_2}$ microlocally near (x^0, ξ^0), $(0,0)$.

Th$^{\underline{m}}$ P of real principal type, $\|u_h\|_{H_h(M)} = O(h^{-N})$

$P_h u_h = f_h \to WF_h(u) \setminus WF_h(f)$ is invariant under the flow of H_{p_0}.

$$P = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \quad \text{rescale } \tilde{P}(x, hD, h) = h^m P.$$

$$\text{and } h^m P = \sum_{|\alpha| \le m} a_\alpha(x)\, h^{m-|\alpha|} (hD)^\alpha.$$

$$\tilde{p}_0(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha.$$

Ie, $\tilde{p}_0(x,\xi) = p_0(x,\xi)$ and is real prin. type!

By Prev Th$^{\underline{m}}$:

$$\Rightarrow WF_h(u) = \underline{\qquad} \sqcup \underbrace{WF(u)}_{\text{without } h}.$$