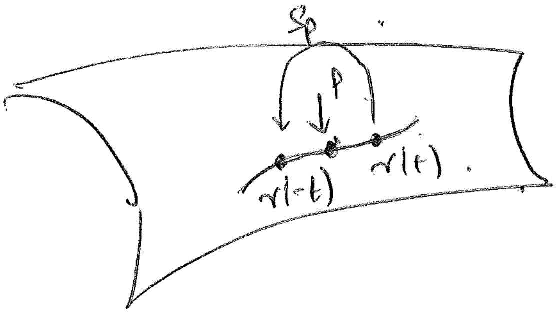
Geometric Fourier Analysis on Riem ~~mflds~~. symm. spaces - Anke Pohl. 07/07/2015.

X - Riem mfld., $p \in X$.

<u>Geodesic symmetry.</u> ~~$s_p(\gamma(t))$~~. γ - geodesic through p. $\gamma(0) = p$.



$$s_p(\gamma(t)) = \gamma(-t).$$

<u>X Riem symm space</u> $\Longleftrightarrow$ For each $p \in X$, s_p is an isometry of X.

1926: E. Cartan

isometry groups.

$$\boxed{\text{Riem symm. spaces}} \longleftrightarrow \boxed{\text{semisimple Lie groups}}$$

$$\updownarrow$$

$$\boxed{\text{semisimple Lie algebras}}$$

de Rham decomp. for Riem. symm. spaces X:
(simple, connected).

$$X \underset{\text{as Riem product}}{\cong} X_{eucl} \times (X_{p,1} \times \dots \times X_{p,m_p}) \times$$

$X_{eucl} \cong \mathbb{R}^k$; $\underbrace{X_{p,1} \times \dots \times X_{p,m_p}}$: irreducible, non-neg. sect. curv.

$$\underbrace{(X_{n,1} \times \dots \times X_{n,m_n})}.$$

irreducible, non-pos. sect. curvature.

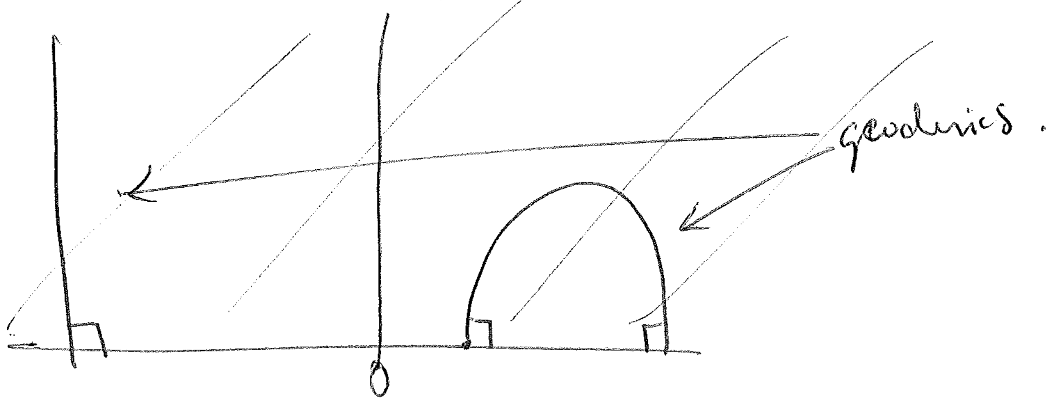
~~irredu~~ __irreducible:__ doesn't split nontrivially into a Riem product.

__From now:__ X irreducible, non pos sect curvature. ($\Rightarrow$ of noncpct type).

__and__ $X \not\cong \mathbb{R}^k$.

__Examples:__ Hyperbolic plane. $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad z = x + iy$$



Connected component of group of Riem isometries.

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm 1\},$$

action by fractional linear transformations.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}.$$

<u>Next thing:</u> Use action coming from group to give a coordinate system.

<u>Note:</u>
* $PSL_2(\mathbb{R})$ is a Lie Group.
* diff elements of $PSL_2(\mathbb{R})$ can act in qualitative different ways.

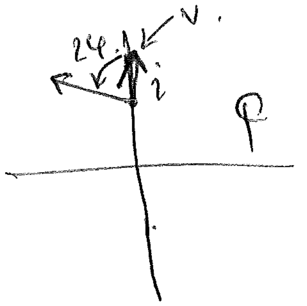
Fix i as origin of $\mathbb{H}$.
fix $\partial_y|_i$ as origin of $S\mathbb{H} = \{v \in T\mathbb{H} : \|v\| = 1\}$.

<u>Classification of elements:</u>

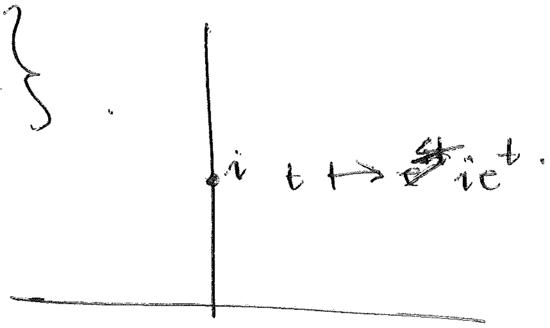
0) for $SL_2(\mathbb{R})$, $M = \{\pm 1\}$ acts trivially on $\mathbb{H}$ and $S\mathbb{H}$.

1) $\mathrm{Stab}_{PSL_2(\mathbb{R})}(i) = \left\{ k_\varphi := \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} : \varphi \in \mathbb{R} \right\} =: K.$

$k_\varphi \cdot i = i$
$k_\varphi \cdot (\partial_y|_i) = $ rotation by 2φ



2) $A = \left\{ a_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} ; t \in \mathbb{R} \right\}$.

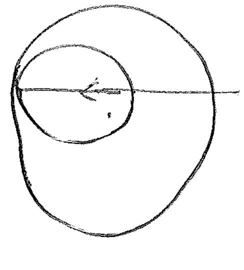


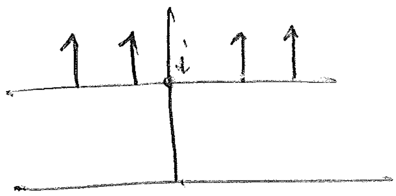
$a_t \cdot i = ie^t$.

$a_t(\partial_y|_i) = e^t \, \partial_y|_{ie^t}$.

A-orbit produces the standard geodesic.

3) $N = \left\{ n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}$



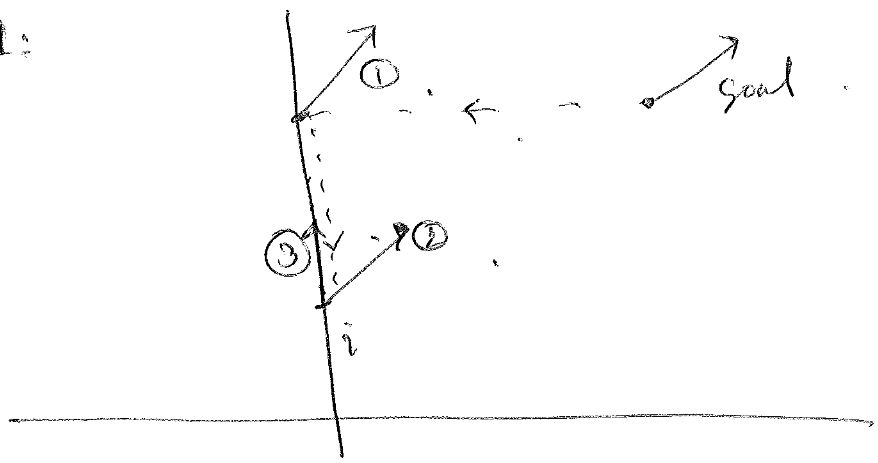


$n_x \cdot i = i + x$.

$n_x \cdot (\partial_y|_i) = \partial_y|_{i+x}$.

Moving of general tangent vectors:

method 1:



~~③ Step ①~~

Step 1 : ③ : use K to produce correct angle.

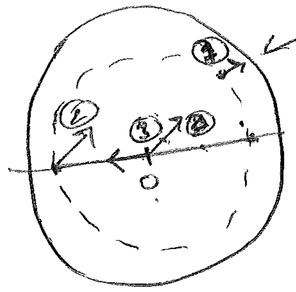
2 : ② : use A to correct height.

3 : ① : use N to slide.

$\Rightarrow$ $G = NAK$. Iwasawa decomposition.
$= PSL_2(\mathbb{R})$. (Gauss algorithm).

By product: $\mathbb{H} = G/K$.

method 2:



← goal. Step 1: ③ : K to rotate.
2: ② : A to bring to correct height. may restrict to $A^+ = \{a_t : t \geq 0\}$.

3: ① : ?

$\Rightarrow$ $G = PSL_2(\mathbb{R}) = K A^+ K$. Cartan decomp. (polar decomposition).

General Riem Symm Space X of noncpct type:

Fix $o \in X$ "origin".

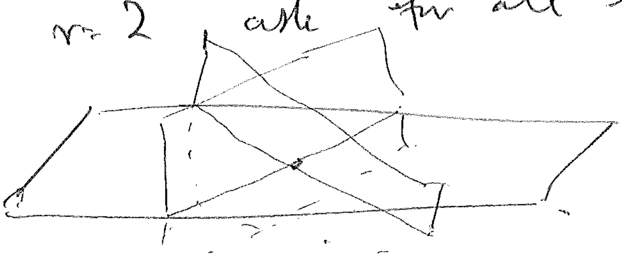
$G := \mathrm{Isom}_o(X)$. Lie gp, ss, finite centre.
connected component.

$K := \mathrm{Stab}_G(o) \leq G$ max cpct.

flat := totally geodesic flat submfld of X of max dimension.

$r := \mathrm{rank}(G) = \mathrm{rank}\, X := \dim(\text{flat})$.

picture $r = 2$, all flats containing o:



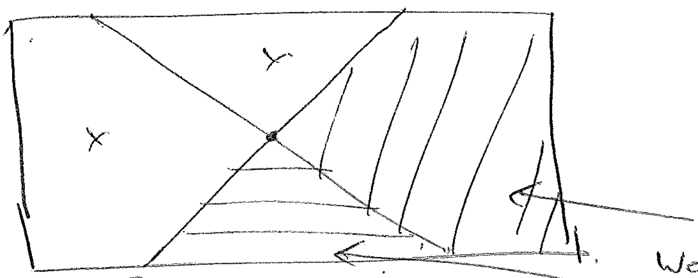
← This is not X. There are flats touching o.

Each flat is an orbit $A \cdot o$ for some r-parameter abelian (noncpct) subgrp A of G of $\mathbb{R}$-diagonalisable elements.

[think: $G = SL_n(\mathbb{R})$, $r = n-1$.

$$A = \left\{ \begin{pmatrix} e^{t_1} & & & \\ & \ddots & & \\ & & e^{t_{n-1}} & \\ & & & e^{-(t_1+\dots+t_{n-1})} \end{pmatrix} : (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1} \right\}.$$

View from top:



Weyl grup $W :=$ generated by reflections at walls.

$= N_K(A) / Z_K(A)$.

normaliser of A in K — centraliser of A in K.

$N_K(A) = \{k \in K : kAk^{-1} = A\}$

$Z_K(A) = \{k \in K : \forall a \in A : kak^{-1} = a\}$

W acts transitively on Weyl chambers.

$A^+ = \{a \in A : a \cdot o \in C_0\}$.

$[A^+ = \{a \in A : t_i > 0\}]$.

$\Rightarrow$ Cartan decomposition $G = K\overline{A^+}K$.

There exists a concept of horocycles which are submanifolds in a certain sense transversal to the flats.

Those containing the origin o are orbits of maximal unipotent subgroups N of G. $\left[\begin{array}{l} G = SL_n(\mathbb{R}) \\ N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\} \end{array}\right.$.

$\Rightarrow$ Iwasawa decomp: $G = NAK$.

Horocycle: orbit of maximal unipot subgroup N. (?).

<u>Fourier Analysis</u>

$\mathbb{D}(X) :=$ algebra of diff ops D of X which are invariant under G.

$\chi : \mathbb{D}(X) \to \mathbb{C}$ homomorphism. | joint eigenvalues

$E_\chi = \{ f \in \mathcal{E}(X) : \forall D \in \mathbb{D}(X) ; Df = \chi(D) f \}$.

<u>Th^m</u> (Harris-Chandra)

The χ's are parametrised by $\mathfrak{a}^*_{\mathbb{C}}/W$ via the Harris-Chandra homomorphism:

$$\mathfrak{a}^*_{\mathbb{C}} \to \{\chi\}.$$

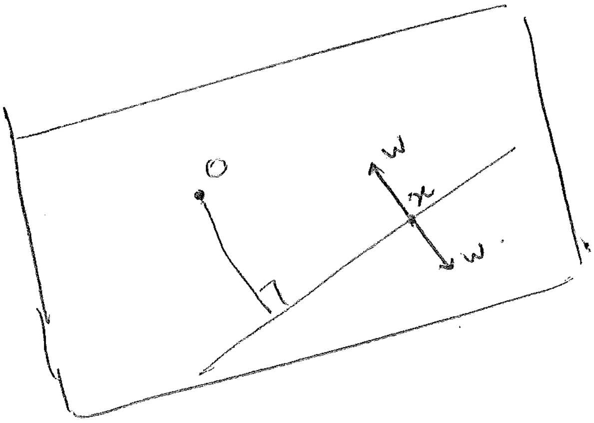
$$\lambda \mapsto \gamma(\cdot)(i\lambda) : \mathbb{D}(X) \to \mathbb{C}.$$

<u>My Rmk</u>: This is a fnct. calcs. $\gamma(D)(i\lambda)$. $D \in \mathbb{D}(X)$.

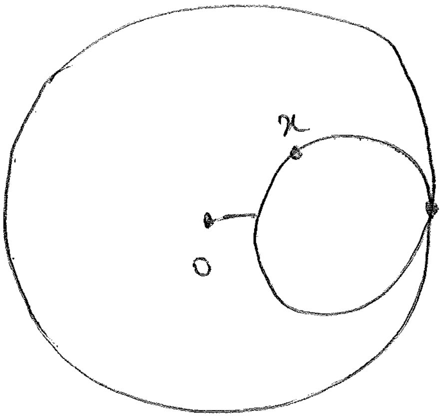
For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$: $\mathcal{E}_{\lambda} := \{ f \in \mathcal{E}(X) : \forall D \in \mathbb{D}(X),$
$Df = \gamma(D)(i\lambda) f \}$.

spectral parameter

Recall FT on $\mathbb{R}^n$: $\tilde{F}(\lambda\omega) := \int_{\mathbb{R}^n} F(x)\, e^{-i\lambda(x,\omega)}\, dx$, $|\omega| = 1$, $\lambda \in \mathbb{R}$.



<u>Here:</u>



$b \in B = K/M$ $\quad M = Z_K(A)$.

$A(x,b) \in \mathfrak{a}$ ($\in$ Lie algebra element).

Remark (for audience guy): $e^{A(x,b)}$ - intersection of horocycle &... what?

elementary waves:

$e_{\lambda,b} : X \to \mathbb{C}, \quad x \mapsto e^{(i\lambda+\rho) A(x,b)}$.

$\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $b \in B$.

$\rho \in \mathfrak{a}^*_{\mathbb{C}}$. specific element. "normalisation"!

$D e_{\lambda,b} = \gamma(D)(i\lambda)\, e_{\lambda,b}$.

Fourier transform. $f: X \to \mathbb{C}$ "nice funct."

$$\underbrace{\tilde{f}(\lambda, b)}_{\substack{\in \\ \mathfrak{a}^*_{\mathbb{C}} \times B}} := \int_X f(x) \cdot e_{-\lambda, b}(x)\, dx.$$

makes sense at least for $f \in \mathcal{D}(X)$.

Inversion formula (Harris Chandra).

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*_{\mathbb{C}} \times B} \tilde{f}(\lambda, b)\, d\!\!\!{}^-\lambda\, db.$$

$$d\!\!\!{}^-\lambda = \frac{d\lambda}{|c(\lambda)|^2}.$$

c – explicitly known. H.C.

Paley - Wiener Thm.

Def: $R > 0$, $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B) \ni \varphi : \mathfrak{a}^*_{\mathbb{C}} \times B \to \mathbb{C} \in C^\infty$:

(I) φ is holomorphic in $\mathfrak{a}^*_{\mathbb{C}}$-variable.

(II) $\forall N \in \mathbb{N}_0$:

$$\varphi(\lambda, b) \ll_N (1 + |\lambda|)^{-N} e^{R \cdot |\mathrm{Im}\,\lambda|}.$$

(think: $\mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}^{\mathrm{rank}(X)}$).

$\mathcal{H}^R_W(\mathfrak{a}^*_{\mathbb{C}} \times B)$ "=" $\{ \varphi \in \mathcal{H}^R : W\text{-inv. in } \mathfrak{a}^*_{\mathbb{C}}\text{-comp} \}$.

$$\mathcal{H}_W := \bigcup_{R>0} \mathcal{H}^R_W.$$

P-W. Thm: $f \mapsto \tilde{f}$ is a bijection

$$\mathcal{D}(X) \to \mathcal{H}_W(\mathfrak{a}^*_{\mathbb{C}} \times B).$$

For any $R > 0$, it restricts to

$$\{f \in \mathcal{D}(X) : \operatorname{supp} f \subset \overline{B_R(0)}\} \to \mathcal{H}^R_W.$$

Plancherel Formula:

$f \mapsto \tilde{f}$ extends to isometry

$$L^2(X) \to L^2(\mathfrak{a}^*_{\mathbb{C}}/W \times B,\ d\lambda\, db).$$

Moreover,

$$\int_X f_1(x)\overline{f_2(x)}\,dx = \frac{1}{|W|}\int_{\mathfrak{a}^*_{\mathbb{C}} \times B} \tilde{f}_1(\lambda, b)\overline{\tilde{f}_2(\lambda, b)}\,d\lambda\,db.$$

$\varphi : X \to \mathbb{C}$ C^∞, $\varphi(0) = 1$, $\varphi(kx) = \varphi(x)$.

spherical $:\Longleftrightarrow$ $\forall D \in \mathbb{D}(X)$: $D\varphi = \lambda_D \varphi$.

(Can lift $\varphi : G \to \mathbb{C}$, φ bi-K-invariant).

Harish-Chandra: spherical functions are parametrized by $\mathfrak{a}^*_{\mathbb{C}}/W$:

$$\varphi_\lambda(g) := \int_K e^{(i\lambda+\rho)(A(kg))}\,dk. \qquad A(x, b),\ gK,\ k$$

$X \times B = G/K \times K/M$, $g \in N\exp(\underbrace{A(g)}_{\in \mathfrak{a}})K$.

Fact: $A(gK, kM) = A(k^{-1}g)$.

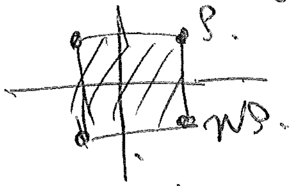
Spherical (Harish-Chandra) transform:

$f: G \to \mathbb{C}$, bi-K-invariant

$$\tilde{f}(\lambda) = \int_G f(g)\,\varphi_{-\lambda}(g)\,dg.$$

<u>Blomer–P:</u> $B \subset A$ bdd, C_ρ convex hull of $W.\rho$.

$\Rightarrow \forall a \in B \ \forall \xi = \lambda + i\eta \in \mathfrak{a}^* + iC_\rho$.



$$\varphi_\xi(k_1 a k_2) = \varphi_\xi(a) \ll_B (1 + |\lambda| \cdot \|a\|)^{-\frac{1}{2}}.$$

$$\left(f \ll g \iff |f| < Cg \quad \exists C > 0 \right)$$