G.F. Savaré   Lecture 4   20/02/2015.

$$I_\varepsilon(u) = \int_0^{+\infty}\left(\frac{\varepsilon}{2}|\dot u|^2 + \varphi(u)\right)d\mu_\varepsilon, \quad u\in AC^2_{loc}, \quad u(0)=\bar u \in D(\varphi).$$

Value function:

$$V_\varepsilon(\bar u) = \min_{\substack{u\in AC^2_{loc}\\ u(0)=\bar u}} I_\varepsilon(u). \qquad \text{(Satisfies dynamic prog. principle).}$$

$$V_\varepsilon(\bar u) = \min\left\{\int_0^T\left(\frac{\varepsilon}{2}|\dot u|^2+\varphi(u)\right)d\mu_\varepsilon + e^{-T/\varepsilon}V_\varepsilon(u(T))\right\}.$$

$$-\frac{d}{dt}V_\varepsilon(u_\varepsilon(t)) = \frac{1}{2}|\dot u|^2_\varepsilon + \frac{1}{2}G_\varepsilon(u_\varepsilon)$$

$$\frac{1}{2}G_\varepsilon^2(u) := \frac{\varphi-V_\varepsilon}{\varepsilon}(u) \quad \left(\text{"="}\ \frac{1}{2}|DV_\varepsilon|^2(u)\right)$$

$$V_\varepsilon(u_\varepsilon(t)) + \frac{1}{2}\int_0^t\left(|\dot u_\varepsilon|^2 + G_\varepsilon^2(u_\varepsilon)\right)dr = V_\varepsilon(\bar u) \qquad (*).$$

Q. How to pass to limit as $\varepsilon\to 0$.

1. $\int_0^T|\dot u_\varepsilon|^2\,dt \le C$, $\int_0^T\varphi(u_\varepsilon)\,dt \le C$.

$\Rightarrow \exists\, u_{\varepsilon_k} \xrightarrow{\text{ptwise}} u$, $|\dot u_\varepsilon| \rightharpoonup v$ in $L^2_{loc}$ $\quad v\ge|\dot u|$

2. $\Gamma\text{-}\lim_{\varepsilon\downarrow 0} V_\varepsilon = \varphi$ $\quad\left[\begin{array}{l} x_n\to x \quad \varepsilon_n\downarrow 0 \\ \liminf_{n\uparrow 0} V_{\varepsilon_n}(x_n)\ge\varphi(x). \\ \lim_{\varepsilon\downarrow 0} V_\varepsilon(x) = \varphi(x).\end{array}\right.$

Taking liminf of $(*)$, obtain (via 1 & 2).

$$\liminf_{\varepsilon\downarrow 0} V_\varepsilon(\bar u) \le \varphi(\bar u)$$

$$\liminf_{\varepsilon\downarrow 0} V_\varepsilon(u_\varepsilon(t)) = \varphi(u(t)).$$

③ $\Gamma\text{-}\liminf_{\varepsilon \downarrow 0} G_\varepsilon(x) := \inf \{ \liminf_{n \downarrow 0} G_{\varepsilon_n}(x_n) : x_n \to x \}$.

$$= G^-(x)$$

Also, $G(x) = \limsup_{\varepsilon \downarrow 0} G_\varepsilon(x) \leq |\partial \varphi|(x)$.

and $|\partial^- \varphi|(x) \leq G^-(x) \leq G(x)$

We also have via Pomere ($\Rightarrow V_\varepsilon(x) \geq -a - 2b\, d^2(x, x_*)$) and Gronwall in general (trivial if $\varphi$ & thus $V_\varepsilon \geq 0$).

$$\int_0^T \frac{1}{2} |\dot{u}|^2 \, dt \leq V_\varepsilon(\bar{u}) - V_\varepsilon(u_\varepsilon(t)) \leq C. \qquad (V_\varepsilon(\bar{u}) \leq \varphi(\bar{u})).$$

$$\left( \varphi(u_\varepsilon) - \frac{\varepsilon}{2} |\dot{u}_\varepsilon|^2 \right)' + |\dot{u}|^2 = 0$$

$V_\varepsilon(u_\varepsilon)$ is the a.c. representative of $\varphi - \frac{\varepsilon}{2} |\dot{u}_\varepsilon|^2$

$$\Rightarrow \varphi(u_\varepsilon) = V_\varepsilon(u_\varepsilon) + \frac{\varepsilon}{2} |\dot{u}_\varepsilon|^2 .$$

$$\Rightarrow \int_0^T \varphi(u_\varepsilon(t)) \, dt \leq C.$$

$W_\varepsilon$ the a.c. representative of $\varphi(u_\varepsilon) - \frac{\varepsilon}{2} |\dot{u}_\varepsilon|^2$.

$$\int_a^b \left( \frac{\varepsilon}{2} |\dot{u}|^2 + \varphi(u_\varepsilon) \right) d\mu_\varepsilon = \int_a^b \left( \varepsilon |\dot{u}|^2 + \left( \varphi - \frac{\varepsilon}{2} |\dot{u}|^2 \right) \right) d\mu_\varepsilon .$$

$$= \int_a^b \left( -\varepsilon W_\varepsilon'(r) + W_\varepsilon(r) \right) d\mu_\varepsilon .$$

$$= W_\varepsilon(a) e^{-a/\varepsilon} - W_\varepsilon(b) e^{-b/\varepsilon}$$

$a = 0, \ b \to +\infty$

$$\Rightarrow \int_0^\infty \left( \frac{\varepsilon}{2} |\dot{u}|^2 + \varphi(u_\varepsilon) \right) d\mu_\varepsilon = W_\varepsilon(0).$$

Back to verifying ① - ③.

Now, for ②; let $x_n \to x$, select subsequence $\varepsilon_m \downarrow 0$

s.t. $\begin{cases} V_{\varepsilon_m}(x_m) = \int_0^{+\infty} \left( \frac{\varepsilon_m}{2} |\dot{u}_n|^2 + \varphi(u_n) \right) d\mu_{\varepsilon_n}. \\ \& \; V_{\varepsilon_m}(x_m) = \liminf V_{\varepsilon_m}(x_m) < +\infty. \end{cases}$

$u_m \to u$ pointwise, $\int_0^T |\dot{u}_m|^2 dt \le C$, $d(u_m(t), x_m) \le C_T \cdot \sqrt{t}$.

put $t = \varepsilon_n s$, then.

$$V_{\varepsilon_m}(x_m) = \int_0^{+\infty} \left[ \frac{1}{2\varepsilon_n} |\dot{u}_n|^2(\varepsilon_n s) + \varphi(u_n(\varepsilon_n s)) \right] e^{-s} ds.$$

$$\geq \int_0^{+\infty} \varphi(u_n(\varepsilon_n s)) e^{-s} ds.$$

$u_m \to x$ so, $d(u_m(\varepsilon_m s), x) \le d(u_m(\varepsilon_m s), x_m) + d(x_m, x)$.

~~and~~ So, Fatou ;.

$$\geq \int_0^{+\infty} \varphi(x) e^{-s} ds = \varphi(x)$$ since $e^{-s} ds$ is probability measure.

Ie, this shows that $\liminf\limits_{n \uparrow 0} V_{\varepsilon_n}(x_n) \geq \varphi(x)$.

③: $I_\varepsilon(u) = \int_0^{+\infty} \left( \frac{\varepsilon}{2} |\dot{u}|^2 + \varphi(u) \right) d\mu_\varepsilon$, $E(t) = \int_0^t |\dot{u}|^2(r) dr$., $E'(t) = |\dot{u}|^2(t)$

Energy of curve.

and $\int \frac{\varepsilon}{2} |\dot{u}|^2 d\mu_\varepsilon = \int \frac{\varepsilon}{2} E'(t) d\mu_\varepsilon = \int_0^{+\infty} \frac{1}{2} E(t) d\mu_\varepsilon$.

$$\Rightarrow \quad = \int_0^{+\infty} \left( \frac{1}{2} E(t) + \varphi(u(t)) \right) d\mu_\varepsilon(t).$$

Use optimal curve $u$, then $V_\varepsilon(x) = I_\varepsilon(u)$, ~~E(t)~~ and

$$E(t) = t \fint_0^t |\dot{u}|^2 dt \underset{\text{Jensen}}{\geq} t \left( \fint |\dot{u}| \right)^2 \geq \frac{d^2(u(0), u(t))}{t}.$$

$$G(x) = \limsup_{\varepsilon \downarrow 0} \frac{\varphi - V_\varepsilon}{\varepsilon}(x).$$

$$V_\varepsilon(x) \geq \int_0^{+\infty} \left[ \frac{d^2(x, u_\varepsilon(t))}{2t} + \varphi(u_\varepsilon(t)) \right] d\mu$$

$$\geq \int_0^{+\infty} Y_t(x)\, d\mu_\varepsilon$$

where $Y_t(x) = \inf_y \frac{d^2(x,y)}{t}$, Yosida approx.

$$\frac{\varphi - V_\varepsilon(x)}{\varepsilon} \leq \int_0^{+\infty} \frac{\varphi(x) - Y_t(x)}{\varepsilon}\, d\mu_\varepsilon(t)$$

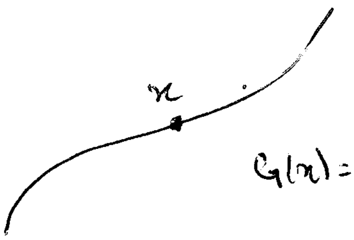
$$\leq \int_0^{+\infty} \frac{\varphi(x) - Y_{s\varepsilon}}{s\varepsilon} (s e^{-s})\, ds$$

Recall $\limsup_{\eta \downarrow 0} \frac{\varphi(x) - Y_\eta(x)}{\eta} = \frac{1}{2} |\partial \varphi|^2(x)$. (Rep slope by Yosida approx)

So, $\limsup_{\varepsilon \downarrow 0} \frac{\varphi - V_\varepsilon}{\varepsilon}(x) \leq \int_0^{+\infty} \frac{1}{2} |\partial\varphi|^2(x)\, s e^{-s}\, ds = \frac{1}{2} |\partial \varphi|^2(x)$.

Ie, $G(x) \leq |\partial \varphi|(x)$.

For reverse (for $G(x)$),




$w \in AC^2_{loc}((a,b);X)$.
$\varphi \circ w \in L^1_{loc}$

$$G(x) = \limsup_{\varepsilon \to 0} \frac{\varphi - V_\varepsilon}{\varepsilon}(x)$$

$$\geq \liminf_{h \downarrow 0} \frac{\varphi(w(t_0)) - \varphi(w_0(t+h))}{h} - \limsup_{h \to 0} \frac{\frac{1}{2}\int_0^{t_0+h} |\dot w|^2 dr}{h}$$

$$= \varphi'(w(t_0)) - \frac{1}{2} |\dot w|^2(t_0)$$

$$\frac{1}{2} G^2(x) \geq \varphi'(w(t_0)) - \frac{1}{2} |\dot w|^2(t_0)$$

But also, $\forall \delta > 0$ $\quad \frac{1}{2} G^2(x) \geq \varphi'(w(t_0))\delta - \frac{1}{2} |\dot w|^2(t_0)\, \delta^2$.

Optimizing $\delta = \dfrac{\varphi'(w(t_0))}{|\dot w|^2(t_0)}$ ?

$$\frac{1}{2} G^2(x) \geq \frac{(\varphi'(w(t_0)))^2}{|\dot w|^2(t_0)}$$

$$|\varphi'(w(t_0))| \leq G(x) \cdot |\dot w|(t_0).$$

Assume that $|\partial^- \varphi| = \overline{\lim}\, |\partial \varphi|$ is an upper gradient for $\varphi$.

Then $G(x) = |\partial^- \varphi|(x)$ is also an upper gradient for $\varphi$.

and the limit function

$$\varphi(u(t)) + \frac{1}{2}\int_0^t |\dot u|^2 + G^2(u(t))\,dt = \varphi(\bar u).$$

$$\frac{d}{dt}\varphi(u(t)) = -|\dot u|^2 = -G^2(u(t)) = -|\partial^- \varphi|(u(t)).$$

In particular, this holds if $\varphi$ is $\lambda$-geodesically convex in $X$. $(|\partial \varphi| = |\partial^- \varphi| = G(x))$.