G.F. Savaré Lecture 3 19/02/2015.

$$I_\varepsilon(u) := \int_0^{+\infty} \left(\frac{\varepsilon}{2} |\dot u|^2 (t) + \varphi(u(t)) \right) d\mu_\varepsilon(t) \qquad u \in AC^2_{loc}([0,\infty); X).$$
$$u(0) = \bar u.$$

If $\bar u \in D(\varphi) \Rightarrow \exists u_\varepsilon$ a minimiser of $I_\varepsilon(u)$.

Inner variation estimate: the map $t \mapsto \varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2} |\dot u_\varepsilon|^2(t) \in W^{1,1}_{loc}([0,\infty))$
and $\frac{d}{dt}\left(\varphi(u_\varepsilon(t)) - \frac{\varepsilon}{2} |\dot u_\varepsilon|^2(t) \right) + |\dot u_\varepsilon(t)|^2 = 0$

In the smooth case, Euler equation
$(\varphi \in C^1, X = \mathbb{R}^d)$ $\quad -\varepsilon u'' + u' + D\varphi(u) = 0,$

$$\Rightarrow \quad -\frac{\varepsilon}{2} \frac{d}{dt} |u'(t)|^2 + |u'(t)|^2 + \frac{d}{dt} \varphi(u(t)) = 0.$$

Pf of inner variation estimate: τ small parameter, $\{R_\tau(t)\}_\tau$
is a smooth family of diffeomorphisms of $(0,+\infty)$, $R_0(t) = t$.
E.g., $R_\tau(t) = t + \tau \xi(t)$, $\xi \in C^\infty_c(0,\infty)$,

$\tilde u_\tau(s) = u_\varepsilon(R_\tau^{-1}(s))$, $t = R_\tau^{-1}(s)$, so $s = R_\tau(t)$ and

$$I_\varepsilon(\tilde u_\tau) = \int_0^{+\infty} \frac{e^{-R_\tau(t)/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{2} \frac{|\dot u_\varepsilon|^2(t)}{R_\tau'(t)} + R_\tau'(t) \varphi(u_\varepsilon(t)) \right).$$

Since u_ε minimiser, $\frac{d}{d\tau} I_\varepsilon(\tilde u_\tau)\Big|_{\tau=0} = 0$, $\partial_\tau R_\tau'(t) = \partial_\tau(1 + \tau \xi'(t)) = \xi'(t)$.

$$\frac{d}{d\tau} I_\varepsilon(\tilde u_\tau) = \int_0^{+\infty} \frac{e^{-t/\varepsilon}}{\varepsilon} \left(-\frac{\varepsilon}{2} |\dot u_\varepsilon|^2 \xi'(t) + \xi'(t) \varphi(u_\varepsilon) \right) dt$$
$$+ \int_0^\infty \left(\frac{-\xi(t)}{\varepsilon} \left(\frac{\varepsilon}{2} |\dot u_\varepsilon|^2 + \varphi(u_\varepsilon) \right) \right) dt.$$

$$= \int_0^{+\infty} \left[\left(\varphi(u_\varepsilon) - \frac{\varepsilon}{2} |\dot{u}_\varepsilon|^2 \right) \underbrace{\left(\xi' - \frac{\xi}{\varepsilon} \right)} + \xi(t) |\dot{u}_\varepsilon|^2 \right] d\mu_\varepsilon = 0 .$$

to pick up distributional derivative!

$$\Rightarrow \frac{d}{dt} \left(\varphi(u_\varepsilon) - \frac{\varepsilon}{2} |\dot{u}_\varepsilon|^2 + |\dot{u}|^2 \right) = 0 .$$

- No Euler equation to play with, so, rephrased WED problem as a (very particular) optimal control problem.

$$V_\varepsilon(x) := \min \{ I_\varepsilon(u) : u \in AC^2_{loc},\ u(0) = x \} .$$

~~Control~~:

Smooth case: $\begin{cases} \dot{u}(t) = v(t) & \text{(velocity)}. \\ u(0) = x \end{cases}$

obtain $u(t; x, v)$ and

$$V_\varepsilon(x) = \min_{v \in L^2_{loc}} \int_0^{+\infty} \left[e^{-t/\varepsilon} \mathcal{L}(u(t; x, v), v) \right] dt ,$$

Lagrangian.

Our Lagrangian $\mathcal{L}(x, v) = \frac{1}{2} |v|^2 + \frac{1}{\varepsilon} \varphi(x)$] control problem.

- Dynamic programming principle:

$$V_\varepsilon(x) = \min_{v \in L^2_{loc}} \left[\int_0^T e^{-t/\varepsilon} \mathcal{L}(u(t; x, v), v) \, dt + V_\varepsilon(u(T; x, v)) e^{-T/\varepsilon} \right] .$$

- Hamilton-Jacobi equation:

$$\frac{V_\varepsilon(x)}{\varepsilon} + H(x, DV(x)) = 0$$

$$H(x, p) := \sup \langle -p, v \rangle - \mathcal{L}(x, v) = \mathcal{L}^*(x, -p).$$

$$\left(= \frac{1}{2} |p|^2 + \frac{\varphi(x)}{\varepsilon} \right) .$$

• Optimal v_*^ε (direction/velocity) providing.

$$H(x, DV_\varepsilon(x)) = \langle DV_\varepsilon(x), v_*^\varepsilon \rangle - \mathcal{L}(x, v_*^\varepsilon).$$

is the initial velocity of the optimal curve $\dot{u}_\varepsilon(0)$.

$$v_*^\varepsilon = \partial_p H(x, -DV_\varepsilon(x)) = \dot{u}_\varepsilon$$

I.e, $\partial_p H(u_\varepsilon(t), -DV_\varepsilon(u(t))) = \dot{u}_\varepsilon(t)$.

$$\boxed{\dot{u}_\varepsilon(t) = -DV_\varepsilon(u_\varepsilon(t)).}$$

Idea: Show that V_ε converges, and that we can pass to limit in $\boxed{}$.

Interpretation: $\varphi - \frac{\varepsilon}{2}|\dot{u}|^2 \overset{?}{=} \varphi(u_\varepsilon) - \frac{\varepsilon}{2}|DV_\varepsilon(u_\varepsilon)|^2$

$$\overset{?}{=} \varphi(u_\varepsilon) - (\varphi - V_\varepsilon) = V_\varepsilon.$$

Hamilton Jacobi true?

$$\frac{d}{dt} V_\varepsilon(u_\varepsilon(t)) = -|\dot{u}_\varepsilon|^2 = -\frac{1}{2}|\dot{u}_\varepsilon|^2 - \frac{1}{2}|\dot{u}_\varepsilon|^2.$$

$$\overset{?}{=} -\frac{1}{2}|\dot{u}_\varepsilon|^2 - |DV_\varepsilon(u_\varepsilon)|^2.$$

$$\text{-HJ} \overset{?}{=} -\frac{1}{2}|\dot{u}_\varepsilon| + \frac{\varphi - V_\varepsilon}{\varepsilon}(u_\varepsilon).$$

$$\boxed{\frac{d}{dt} V_\varepsilon(u_\varepsilon(t)) = -\frac{1}{2}|\dot{u}_\varepsilon|^2 - \frac{\varphi - V_\varepsilon}{\varepsilon}(u_\varepsilon(t))}$$

DeGiorgi metric formulation of the GF of V_ε.

$$V_\varepsilon(u_\varepsilon(T)) + \frac{1}{2}\int_0^T \left[|\dot{u}_\varepsilon|^2 + \frac{\varphi - V_\varepsilon}{\varepsilon}(u_\varepsilon(t))\right] dt = V_\varepsilon(\bar{u})$$

? ⇓ ? ⇓ ? ⇓ ? ⇓ ← As $\varepsilon \to 0$.

$$\varphi(u(T)) + \frac{1}{2}\int_0^T \left[|\dot{u}|^2 + G(u(t))\right] dt \le \varphi(\bar{u}).$$

upper gradient

Enough, easier to prove.

Minimising movement.

Yosida approximation of φ:

$$Y_t(x) := \inf_{y \in X} \frac{1}{2t} d^2(y,x) + \varphi(y).$$

$$Y_t \uparrow \varphi, \quad V_\varepsilon(x) = \inf \int_0^{+\infty} \frac{e^{-t/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{2} |\dot{u}|^2 + \varphi(u) \right) dt, \; V_\varepsilon \uparrow \varphi \text{ as } \varepsilon \to 0.$$

$$\begin{cases} \partial_t Y_t(x) + \frac{1}{2} |DY_t|^2(x) = 0. \\ Y_0(x) = \varphi(x). \end{cases}$$

Properties of V_ε:

- V_ε is l.s.c: if $x_n \to x$, $\liminf_{n\to\infty} V_\varepsilon(x_n) \geq V(x)$.

  It follows by l.s.c. of $I_\varepsilon(u)$, by writing

  $V_\varepsilon(x_m) = I_\varepsilon(u_m)$ (where u_m minimises $u_m(0) = x_m$).

  and by before, $\exists\, u_{m_k} \to u_\varepsilon$ ptwise. and

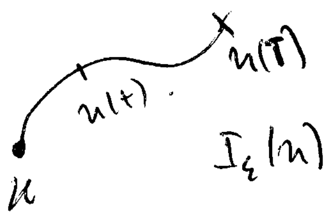
  u is a minimum of I_ε with initial value x_ε.

- If $\varepsilon_1 < \varepsilon_2$, then $V_{\varepsilon_1}(x) \geq V_{\varepsilon_2}(x)$ since,

$$V_\varepsilon(x) = \min_{\substack{u \in AC_{loc} \\ u(0) = x}} \int_0^{+\infty} e^{-t} \left(\frac{1}{2\varepsilon} |\dot{u}|^2 + \varphi(u(t)) \right) dt. \quad \left(\text{change of variables}\right)$$

optimal.

$$w_\varepsilon(t) = u_\varepsilon(\varepsilon t).$$

- 

$$I_\varepsilon(u) := \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{2} |\dot{u}|^2 + \varphi(u(t)) \right) dt \quad \text{I}$$

$$+ e^{-T/\varepsilon} \int_0^\infty \left[\frac{e^{-t/\varepsilon}}{\varepsilon} \left(|\dot{u}|(t+T) + \varphi(u(t+T)) \right) \right]$$

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$$= \quad ① + e^{-T/\varepsilon} V_\varepsilon(u(T)).$$

So,

$$V_\varepsilon(x) \leq \int_0^T \frac{e^{-t/\varepsilon}}{\varepsilon} \left(\frac{\varepsilon}{2} |\dot u|^2 + \varphi(u) \right) dt + e^{-T/\varepsilon} V_\varepsilon(u(T))$$

$$u \in AC^2_{loc}, \quad u(0) = x.$$

equality for optimal u_ε, so,

$$V_\varepsilon(x) = \min_{\substack{u \in AC^2_{loc} \\ u(0)=x}} \int_0^T \left(\frac{\varepsilon}{2} |\dot u|^2 + \varphi(u) \right) d\mu_\varepsilon + V_\varepsilon(u(T))$$

<u>Regularity of V_ε along AC curves, with integrable φ:</u>

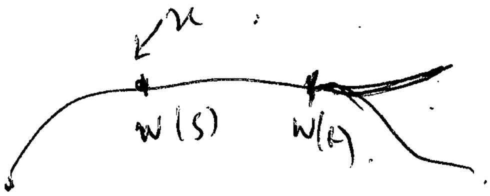
Suppose that $w \in AC^2_{loc}((a,b); X)$ with $\varphi \circ w \in L^1_{loc}(a,b)$.

Then, $t \mapsto V_\varepsilon(w(t))$ is in $AC_{loc}((a,b))$ and $\cancel{\left|\frac{d}{dt} V_\varepsilon(w(t))\right| \leq}$

$$\left| \frac{d}{dt} V_\varepsilon(w(t)) \right| \leq |\dot w|(t) \sqrt{\frac{\varphi_\varepsilon(w(t)) - V_\varepsilon(w(t))}{\varepsilon}}.$$

upper gradient for V_ε.

Dynamic programming principle



$$V_\varepsilon(w(s)) - V_\varepsilon(w(t)) \leq \int_s^t \left(\frac{\varepsilon}{2} |\dot w|(r) + \varphi(w(r)) \right) \frac{e^{-\frac{(t-s)}{\varepsilon}}}{\varepsilon} dr$$

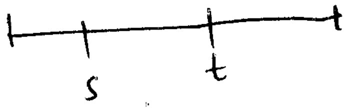
L^1 function integrated s to t.

$$+ \underbrace{\left(e^{-(t-s)/\varepsilon} - 1 \right) V_\varepsilon(w(t))}_{\text{Lipschitz}}.$$

reverse order, get same estimate.

Which proves $t \mapsto V_\varepsilon(w(t))$ $AC_{loc}((a,b))$.

$w = u_\varepsilon$; the minimizer of I_ε, with initial datum x.



$$e^{-s/\varepsilon} V_\varepsilon(u_\varepsilon(s)) = \int_s^t \left(\frac{\varepsilon}{2}|\dot u_\varepsilon|^2 + \varphi(u_\varepsilon)\right)\frac{e^{-r/\varepsilon}}{\varepsilon}\,dr + V_\varepsilon(u_\varepsilon(t))e^{-t/\varepsilon}.$$

We know V_ε along u_ε is AC. Choose s a Lebesgue point for $|\dot u_\varepsilon|^2$ and for $\varphi(u_\varepsilon)$.

$$\frac{V_\varepsilon(u_\varepsilon(s))\,e^{-s/\varepsilon} - V_\varepsilon(u_\varepsilon(t))\,e^{-t/\varepsilon}}{s-t} = \underbrace{\frac{1}{t-s}\int_s^t \left(\;\text{———}\; \right) e^{-r/\varepsilon}\,dr}_{\text{Lebesgue theorem}}$$

Since s is Lebesgue point, $t \to s$ limit exists.

$$-\frac{d}{dt}\Big|_{t=s}\left(V_\varepsilon(u_\varepsilon)\,e^{-t/\varepsilon}\right) = \left(\frac{\varepsilon}{2}|\dot u_\varepsilon|^2(s) + \varphi(u_\varepsilon(s))\right)\frac{e^{-s/\varepsilon}}{\varepsilon}$$

$$= -e^{-s/\varepsilon}\left[V'(u_\varepsilon(s)) - \frac{1}{\varepsilon}V_\varepsilon(u_\varepsilon(s))\right]$$

and so, $$-\frac{d}{dt}\Big|_{t=s}\left(V_\varepsilon(u_\varepsilon)\right) = \frac{1}{2}|\dot u_\varepsilon|^2(s) + \frac{\varphi(u_\varepsilon(s)) - V_\varepsilon(u_\varepsilon(s))}{\varepsilon}.$$

which is the crucial formula.