G.F. - Savaré - Lecture 2 18/02/2015.

(X,d) complete metric space, $\varphi : X \to (-\infty,+\infty]$ l.s.c.

$\varphi(u) \geq -a - b\, d^2(u, u_*)$, $a,b \geq 0$, $u_* \in X$.

$$I_\varepsilon(u) := \int_0^{+\infty} \left(\frac{\varepsilon}{2}|\dot u|^2 + \varphi(u)\right) d\mu_\varepsilon(t), \quad \mu_\varepsilon = \frac{e^{-t/\varepsilon}}{\varepsilon}\mathcal{L}^1.$$

$u \in AC^2_{loc}([0,+\infty); X)$, $u(0) = \bar u$.

Existence ① I_ε is l.s.c with respect to ptwise convergence.

② If $(u_n)_n \in AC^2_{loc}([0,+\infty); X)$, $u_n(0)$ is bounded, $I_\varepsilon(u_n) \leq C < +\infty$, then $\exists u_{n_k} \to u$ ptwise. $\exists u \in AC^2_{loc}([0,+\infty); X)$.

①+② => Existence of minimizer.

Rmk. In addition to assumptions above, need $I_\varepsilon(u) < \infty$ for at least one u. But, if $\bar u \in D(\varphi)$ one can always choose $u(t) = \bar u$, $|\dot u| = 0$, $I_\varepsilon(u) = \int_0^{+\infty} \varphi(\bar u)\, d\mu_\varepsilon = \varphi(\bar u) < \infty$.

② is compactness. In gen. metric space, you won't have this. We can obtain ② if X is locally compact. but really, $\varphi(\bar u) < \infty$ should really provide compactness:

If $(u_n)_n \subset X$, bdd, separable, $\varphi(u_n) \leq C$ then $\exists$ convergent subsequence.

Compactness topology: σ is a topology in X, weaker than the distance

(1) d is σ-l.s.c. 2) $(u_n)_n \subset X$ is bdd, $\varphi(u_n) \leq C$.
$\Rightarrow \exists u_{n_k} \to u$ in σ, $\varphi(u) \leq C$.

(3) For every $x \in X$, $\mathcal{U}$ σ-nbh of x, $\exists V \ni x$, V σ-open $\exists \delta > 0$: $\mathcal{U} \supset B_\delta(y)$ $\forall y \in V$

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Example (1) X Hilbert or Banach, σ = weak topology

(2) $X = (\mathcal{P}_2(Y), W_2)$ Wasserstein space, σ weak top.

(3) σ-induced by another distance, $\tilde{d}(x,y) \le c\, d(x,y)$.

When we have such a σ-top, ① and ② convergence is w.r.t. σ.

<u>Global Poincaré inequality for μ_ε:</u>

Let $w \in AC^2_{loc}([0,+\infty);\mathbb{R})$ with $\int_0^{\infty} |w'(t)|^2 \, d\mu_\varepsilon(t) < \infty$; $w(0)=0$

Then $\frac{1}{4\varepsilon^2}\int_0^{+\infty} |w(t)|^2 \, d\mu_\varepsilon(t) \le \int_0^{+\infty} |w'(t)|^2 \, d\mu_\varepsilon(t)$.

<u>Rmk</u> $\int_0^{\infty} |w'(t)|^2 \, d\mu_\varepsilon(t) < \infty \Rightarrow w \in W^{1,2}(\mathbb{R}, d\mu_\varepsilon)$.

In fact, Poincaré inequality improves to $+\infty$ replaced by $T \in (0,+\infty]$

<u>Pf</u> $\varepsilon = 1$ suffices: $\varepsilon^2 \int_0^{+\infty} |w'(t)|^2 \frac{e^{-t/\varepsilon}}{\varepsilon} dt$. $(t/\varepsilon = s, \ t = \varepsilon s)$.

$$= \int_0^{+\infty} e^{-s} \varepsilon^2 |w'(\varepsilon s)|^2 ds = \int_0^{+\infty} e^{-s} |\tilde{w}'(\varepsilon s)|^2 ds.$$

$$\overset{?}{\ge} \frac{1}{4}\int_0^{+\infty} e^{-s} |\tilde{w}(s)|^2 ds = \frac{1}{4}\int_0^{+\infty} \frac{e^{-t/\varepsilon}}{\varepsilon} |w(t)|^2 dt.$$

So, we prove $\forall T > 0$ $\int_0^T |w'(t)| e^{-t} dt \ge \frac{1}{4}\int_0^T |w(t)|^2 e^{-t} dt$.

$f(t) := e^{-t/2} w(t)$, $w(t) = e^{t/2} f(t)$, $w'(t) = e^{t/2} f'(t) + \frac{1}{2} e^{t/2} f(t)$.

$$\int_0^T |f' + \tfrac{1}{2} f|^2 dt = \int_0^T \left(|f'(t)|^2 + \tfrac{1}{4}|f(t)|^2 + f(t) f'(t)\right) dt.$$

$$= \int_0^T |f'(t)|^2 + \tfrac{1}{4} |f(t)|^2 dt + \tfrac{1}{2} f(T)^2 \cancel{- \tfrac{1}{2} f(0)}$$

0 since $w(0)=0$

neglect pos term $\rightarrow$ $\ge \frac{1}{4}\int_0^T |f(t)|^2 dt$. $\blacksquare$

Rmk. Choose $\lambda < \frac{1}{4\varepsilon^2}$, $w \longmapsto \int_0^{+\infty} \left(|w'(t)|^2 - \lambda |w(t)|^2 \right) d\mu_\varepsilon$ is lower semicontinuous w.r.t., say, ~~any topology~~ w.r.t. weak convergence in L^2_{loc} or pointwise convergence.

Rmk. $\varphi(u) \geq -a - b\, d^2(u, u_*) \geq -a - b\, d^2(u, \bar{u}) - 2b\, d^2(\bar{u}, u_*)$.

$$=: A - B\, d^2(u, \bar{u}).$$

$$\left(A = a + 2b d^2(u, u_*), \quad B = 2b. \right).$$

If $u \in AC_{loc}([0, +\infty); X)$, $L(t) := \int_0^t |\dot{u}|(s)\, ds \geq d(u(t), \bar{u})$.

Moreover, $L \in AC_{loc}([0, +\infty); X)$ and $L'(t) = |\dot{u}|(t)$.

So,

$$I_\varepsilon(u) = \underbrace{\int_0^{+\infty} \overbrace{\frac{\varepsilon}{2} |\dot{u}|}^{= \frac{\varepsilon}{2} \dot{L}^2(t)} d\mu_\varepsilon - \lambda \int_0^{+\infty} \frac{\varepsilon}{2} L^2(t)\, d\mu_\varepsilon}_{\mathcal{B} =} + \int_0^{+\infty} \Big(\underbrace{\frac{\lambda\varepsilon}{2} L^2(t) + \varphi(u(t))}_{\geq B d(u(t), \bar{u})} + A \Big)\, d\mu_\varepsilon - A.$$

by Poincaré, positive if $\frac{1}{16\varepsilon^2} = \lambda < \frac{1}{4\varepsilon^2}$, $\frac{\lambda\varepsilon}{2} = \frac{1}{16\varepsilon} > B$.

I.e., by choosing ε small enough, $I_\varepsilon(u)$ well defined, positive, finite.

$$\mathcal{B} \geq \int_0^{+\infty} \frac{\varepsilon}{4} |\dot{u}|^2\, d\mu_\varepsilon \quad \text{by Poincaré}.$$

This is why we have $\int_0^{+\infty} \frac{\lambda\varepsilon}{2} L^2(t)\, d\mu_\varepsilon \geq B d^2(u(t), \bar{u})$.

u_n pointwise converging to u; (I) $I_\varepsilon(u_n) \leq C < +\infty$. $\forall n$.

(II) $\int_0^{+\infty} |\dot{u}_n|^2\, d\mu_\varepsilon \leq C'$ $\quad \forall n$.

(III) $\dot{L}_n$ is bdd in $L^2_{loc}(0, +\infty)$

(IV) $\dot{L}_{n_k} \rightharpoonup \dot{L}$ in $L^2_{loc}(0,+\infty)$

$L_n \to L$ pointwise.

$\dot{L} \geq |\dot{u}|$ since $d(u_m(t), u_m(s)) \leq \int_s^t \dot{L}_n(r)\,dr$.

$$\Downarrow$$

$$d(u(t), u(s)) \leq \int_s^t \dot{L}(r)\,dr.$$

Then,

$$\liminf_{n\to+\infty} I_\varepsilon(u_n) \geq \int_0^{+\infty} \left(\tfrac{\varepsilon}{2} |\dot{L}|^2(t) - \tfrac{\lambda\varepsilon}{2} L^2(t)\right) d\mu_\varepsilon(t).$$

$$+ \int_0^{+\infty} \left(\tfrac{\lambda\varepsilon}{2} L^2(t) + \varphi(u(t)) + A\right) d\mu_\varepsilon(t) - A.$$

By Fatou lemma.

$$= \int_0^{+\infty} \left(\tfrac{\varepsilon}{2} |\dot{L}|^2(t) + \varphi(u(t))\right) d\mu_\varepsilon(t).$$

$$\geq \int_0^{+\infty} \left(\tfrac{\varepsilon}{2} |\dot{u}|^2(t) + \varphi(u(t))\right) d\mu_\varepsilon(t)$$

Then, I_ε l.s.c.

<u>Compactness</u> $\to$ Equi-continuity of $(u_n)_n$.

$\to$ $(u_n(t))_n \subset K$, K compact.

u_n sequence, $u_n(0)$ is bdd, $I_\varepsilon(u_n) \leq C$.

$$\int_0^T |\dot{u}_n|^2\,dt < C_T \Rightarrow d(u_n(t), u_n(s)) < \int_s^t |\dot{u}_n|(r)\,dr$$

$$\overset{\text{Hölder}}{\leq} \sqrt{|t-s|}\left(\int_s^t |\dot{u}_n|^2(r)\,dr\right)^{\frac{1}{2}}$$

$$\leq \sqrt{C_T |t-s|}$$

Then (u_n) is uniformly $\frac{1}{2}$-Hölder continuous in any finite interval $[0,T]$.

$\int_0^T \varphi_t(u_n(t))\,dt \le C$, (not pt-wise, but L^1 bound).

Aubin-Lions-Simon Th^{m}: in Hilbert Space, this is enough.
In metric setting, this is not enough, need to change
Arzela-Ascoli.

<u>Strategy</u>: find dense, countable set of times $D \subset [0,T]$ s.t.
$(u_n(t))_n$ is converging (upto subsequence) $\forall t \in D$.

Fix a countable base $\mathcal{J}$ of open intervals of $(0,T)$.
$\mathcal{J} = \{\underset{I_k}{\underbrace{(a_k, b_k)}} \subset (0,T)\}$.

$$C \ge \int_{a_k}^{b_k} \varphi_t(u_n(t))\,dt \quad \text{and} \quad C \ge \int_{a_k}^{b_k} \liminf_{n\to\infty} \varphi_t(u_n(t))\,dt.$$

$\Rightarrow \exists\, t_k \in (a_k, b_k)$: $\liminf_{n\nearrow+\infty} \varphi_t(u_n(t_k)) < +\infty$.

$\Rightarrow \exists$ subsequence $\varphi(u_{n_k}(t_k)) \le C$. So
$u_{n_k}(t_k) \to w(t_k)$.

Proceed by induction to find $t_k \in (a_k, b_k)$ and
by diagonal argument, u_n : $u_{n_h}(t_k) \xrightarrow{h\uparrow\infty} w(t_k)$. $\forall k$.

$D = \{t_k \in (a_k, b_k)\}$ is dense in $[0,T]$.

$$d(w(t_k), w(t_j)) \le \liminf_{h\uparrow\infty} d(u_{n_h}(t_k), u_{n_h}(t_j)) \le \sqrt{C|t_k - t_j|}.$$

$\Rightarrow$ w extends to a Hölder cts function.