Weighted Energy Dissipation method for gradient flows - Savaré. Lecture 1. 17/02/2015.

Gradient flows. X ambient space, $\varphi : X \to (-\infty, +\infty]$ functional.

In specific example: $X = \mathbb{R}^n$, $\varphi \in C^1$, then G.f. is solving.

$$\begin{cases} \dot{u}(t) + D\varphi(u(t)) = 0 \\ u(0) = u_0 \end{cases}$$

- $X = H$, Hilbert space, φ is convex (or λ-convex) + lower s.c.

$D\varphi \rightsquigarrow \partial\varphi$ (~~multi~~ valued, sub differential).

$$\begin{cases} \dot{u}(t) + \partial\varphi(u(t)) \ni 0 \\ u(0) = u_0 \end{cases}$$

Brezis, Crandall, Komura, Pazy. '70.

- $X = B$ space, φ is convex + C^1; φ is diff of convex functions $\Big\}$ Requires refined techniques
- $X =$ metric space.

De Giorgi, Marino, Saccon, Tosques, De Giovanni ~ '90.

Ambrosio - Gigli - S.

Banach Spaces. $\partial\varphi(u(t)) \in X'$ (dual $\to$ in finite dim or Hilbert, identify X' with H).

$$J\dot{u}(t) + \partial\varphi(u(t)) \ni 0 .$$

dual inclusion. $\frac{1}{2}\|u\|_X^2$. (Colli; Visintin, Colliv.)

(X,d) complete metric space, $\varphi: X \to (-\infty, +\infty]$ l.s.c.

In smooth situation,

$$\dot{u}(t) + D\varphi(u(t)) = 0 \iff \frac{d}{dt}\varphi(u(t)) = -|\dot{u}(t)|^2 = -|D\varphi(u(t))|^2$$

vector eq^n.

$$\iff \underbrace{\frac{d}{dt}\varphi(u(t)) + \frac{1}{2}\left(|\dot{u}(t)|^2 + |D\varphi(u(t))|^2\right) \le 0}_{\text{Scalar eq}^n.}$$

Chain rule:

$$\frac{d}{dt}\varphi(u(t)) = \langle D\varphi(u(t)), \dot{u}(t)\rangle = -|\dot{u}(t)|^2 = -|D\varphi(u(t))|^2.$$

or by C-S inequality as above!

integration.

$$\iff \varphi(u(t)) + \frac{1}{2}\int_0^t \left(|\dot{u}(r)|^2 + |D\varphi(u(r))|^2\right)dr \le \varphi(u_0)$$

$\forall t \ge 0.$

Note, in B-space,

$\langle D\varphi(u(t)), \dot{u}(t)\rangle$. in duality, $X^* \times X$,

So, $\langle D\varphi(u(t)), \dot{u}(t)\rangle = -\|\dot{u}(t)\|_X = -\|D\varphi(u(t))\|_{X^*}$.

<u>Back to (X,d) metric space:</u>

$u: I \to X$ in AC^p ($u \in AC^p(I,X)$). if $\exists\, v \in L^p(I)$, $v \ge 0$.

$$d(u(s), u(t)) \le \int_s^t v(r)\,dr \quad s,t \in I,\ s<t. \quad (*).$$

$p \in [1,\infty]$. — Yes, $p=\infty$ is ok., $AC = AC^1$.

If $u \in AC(I,X)$, then for almost every $t \in I$,

$$\exists \lim_{h\to 0} \frac{d(u(t), u(t+h))}{|h|} =: |\dot{u}|(t)$$

↑ Borel function.

Thm. Theorem.

$|\dot u| \leq v \Rightarrow |\dot u| \in L^p$ and $|\dot u|$ is the minimal v. satisfying (*)

If X is Hilbert or Banach reflexive,

$\Rightarrow u \in AC(I, X)$

$\Rightarrow u$ is a.e. differentiable and

$$\exists \dot u(t) = \lim_{h \to 0} \frac{u(t+h)-u(t)}{h}.$$

- $|\dot u|(t) = \|\dot u(t)\|_X$.
- $u(t) = u(s) + \int_s^t \dot u(r)\,dr$.

Upper gradient and slope

$G: X \to [0,\infty]$ is an upper gradient for φ if for every $u \in AC([0,1], X)$ s.t.

$$\text{(UG1)} \quad \int_0^1 G(u(t))\,|\dot u|(t)\,dt < +\infty$$

~~then,~~

$$\text{(UG2)} \quad |\varphi(u(1)) - \varphi(u(0))| \leq \int_0^1 G(u(t))\,|\dot u|(t)\,dt.$$

- Note, in B-space.

$$\left|\frac{d}{dt}\varphi(u(t))\right| = |\langle D\varphi(u(t)), \dot u(t)\rangle| \leq \|D\varphi(u(t))\|\,\|\dot u(t)\|.$$

In this situation replace $D\varphi$ by G.

- Upper Grad. ~~implies~~ (UG1) $\Rightarrow \left|\frac{d}{dt}\varphi(u(t))\right| \leq G(u(t))\,|\dot u(t)|$ a.e. (UG3)

- Candidate for Upper G: Slope (descending).

$$|\partial\varphi|(u) := \limsup_{v \to u} \frac{(\varphi(u) - \varphi(v))_+}{d(u,v)} \qquad (x)_+ = \text{positive part of } x.$$

Remark. If $\varphi(u(t))$ differentiable at t, then.

$$\left|\frac{d}{dt}\varphi(u(t))\right| \le |\partial\varphi|(u(t))\,|\dot{u}|(t).$$

I.e, slope is always a candidate "upper gradient". (UG1)
However, you don't always get (UG2).

When $|\partial\varphi|$ is an upper gradient, then

$$|\varphi(u(t))-\varphi(u(s))| \le \int_s^t |\partial\varphi|(u(r))\,|\dot{u}|(r)\,dr.$$

Let (X,d) be a complete metric space, $\varphi: X \to (-\infty,+\infty]$ be a l.s.c. functional, $G: X \to [0,+\infty]$ an upper gradient. Given $u_0 \in X$, a curve $u: [0,+\infty) \to X$ is a G.F. of φ starting from u_0 if $u \in AC^2_{loc}([0,+\infty);X)$ and

$$\varphi(u(t)) + \frac{1}{2}\int_0^t \left(|\dot{u}|^2(r) + G^2(u(r))\right)dr \le \varphi(u_0) \quad \forall t \ge 0.$$

$$D(\varphi) = \{u \in X : \varphi(u) < +\infty\}.$$

- From the def^n and the UG property,

$$\varphi\circ u \in AC_{loc}([0,+\infty]) \text{ and}$$
$$\frac{d}{dt}\varphi(u(t)) = -|\dot{u}|^2(t) = -G^2(u(t)) \quad t\text{-a.e.}$$

- Assume that: I $\varphi(u) \ge -a - b\,d(u,u_*)$ for some $a,b \ge 0$, $u_* \in X$.
  II If $(u_n) \in X$; $d(u_m,u_n) \le C$, $\varphi(u_m) \le C$ $\forall n,m \in \mathbb{N}$.
  $\Rightarrow \exists\, u_{n_k} \to u$.
  (III) $G\ (\le |\partial\varphi|)$ is lower semicontinuous.

Then (I)–(III)

$\Rightarrow \forall \bar{u} \in D(\varphi)$, $\exists\, u \in AC^2_{loc}([0,+\infty), X)$. a G.F. for φ starting at $\bar{u}$.

Variational WED formulation:

$\varepsilon > 0$, $\mu_\varepsilon = \frac{e^{-t/\varepsilon}}{\varepsilon}\mathcal{L}^1$, probability measure on $[0,+\infty)$.

$$I_\varepsilon(u) := \int_0^{+\infty} \left(\frac{\varepsilon}{2}|\dot{u}|^2(t) + \varphi(u(t))\right) d\mu_\varepsilon(t).$$

$$= \int_0^{+\infty} e^{-t/\varepsilon}\left(\frac{1}{2}|\dot{u}|(t) + \frac{1}{\varepsilon}\varphi(u(t))\right) dt. \qquad u \in AC^2_{loc}([0,+\infty);X)$$

- Let u_ε be a minimiser of I_ε among all curves in $AC^2_{loc}([0,+\infty);X)$ with $u(0) = \bar{u}$. (u_ε exists assum (I) & (II))

- $\exists\, \varepsilon_n \downarrow 0$. $u_{\varepsilon_n}(t) \to u(t)$. $\forall t$.
- u is a gradient flow for φ (and G) (you need 3)

lower semicontinuity of G.

History:

φ: quadratic, X - Hilbert $\to$ Lions–Magenes.

Ilmanen '94, De Giorgi '96. (Minimal surfaces).

Mielke, Ortiz, Stefanelli – Hilbert case, φ convex.

Smooth (finite dim case): u_ε is a minimiser.

$$\Rightarrow \underbrace{-\varepsilon u''_\varepsilon(t)}_{\text{grad flow perturbed by 2nd order term.}} + u'_\varepsilon(t) + D\varphi(u_\varepsilon(t)) = 0. \qquad (**)$$

(**) ~~This~~ is a elliptic problem, because we have $-\varepsilon u''(t)$, ie, sign is $-$ve.

Fix $v \in C_c^\infty([0,\infty);\mathbb{R}^d)$, $u+\delta v$, δ small.

$I_\varepsilon(u_\varepsilon+\delta v) \geq I_\varepsilon(u_\varepsilon)$, ie check

$$\frac{\partial}{\partial \delta}\int_0^{+\infty}\left(\frac{\varepsilon}{2}|\dot u_\varepsilon+\delta\dot v|^2+\varphi(u_\varepsilon+\delta v)\right)d\mu_\varepsilon\Big|_{\delta=0}$$

$$= \int_0^{+\infty}\varepsilon\langle \dot u_\varepsilon(t),\dot v(t)\rangle + \langle D\varphi(u_\varepsilon(t)), v(t)\rangle\, d\mu_\varepsilon \quad \forall v\in C_c^\infty([0,+\infty),\mathbb{R}^d)$$

Integration by parts: $\int_0^{+\infty}\varepsilon\langle w(t),\dot v(t)\rangle d\mu_\varepsilon = \int_0^{+\infty}\langle -\varepsilon\dot w(t)+w(t), v(t)\rangle d\mu_\varepsilon$ $+ \underbrace{\varepsilon\langle w(0),v(0)\rangle}$

~~if $w \notin C_c^\infty$~~

Since $0\in$ spt w.
in gen,

So, $\int_0^{+\infty}\langle -\varepsilon(\underbrace{u''_\varepsilon+u'_\varepsilon}+D\varphi(u)), v(t)\rangle\, d\mu_\varepsilon(t)=0$

comes for setting $w=\dot u_\varepsilon$

Why gen upper gradient?
$\partial\varphi$ happens up:

Marginal functionals, X = Hilbert, not convex or λ convex
$\varphi(u)=\min \Phi(u,x)$, $x\in Y\leftarrow$ some space Y, Φ nice func.

$X = H^{-1}(\Omega)$, $x\in H_0^1(\Omega)$, $\Phi(u;x) = \int_\Omega$ ~~$W(x)$~~ $dx + \frac{1}{2}\int|u-x|^2 dx + \frac{1}{2}\int_\Omega|Dx|^2dx$ $+\int_\Omega W(x)\,dx$.

Candidate diff. $\partial\varphi(u) = -\Delta(u-x)$, where x solves.
$-\Delta x + W'(x) + x = u$ $\leftarrow$ more than one sol^n.
Sub diff multi-valued!

$\leftarrow W$

~~Hmmm~~

The equation to solve is

$$\begin{cases} \partial_t u - \Delta(u-\chi) = 0 \\ -\Delta\chi + W'(\chi) + \chi = u \end{cases}$$

So, Selgue is not good, because if we take u_t minimizer, it may flip b/w solns to b/c $\partial\varphi$ is multi-valued.

~~Alternatively~~ G.F.: $\varphi(u) = \inf_{\chi} \left(E(u) + u\chi + G(\chi) \right)$

$= E(u) - G^*(u)$.

and $\begin{cases} \partial_t u + DE(u) = \chi \\ DG(\chi) = u \end{cases}$.

~~Can't be solved via G.F. machinery.~~

~~Idea: this choice of G is the "right" G, it is choosing minimum of the χ's.~~

$$\partial_t u + DE(u) + \underbrace{(DG)^{-1}}(u) = 0$$

If not G.F., ie convex then this is multi valued, which is the problem in prev. example.