N. Gigli Lecture 5 26/02/2015.

Differentiable structures on M.M. Spaces.

F.O.D. Structure.

(X, d, m), complete, separable, m non-neg Radon.

L^2-normed L^∞-module. $(M, \|\cdot\|_M)$.

(I) multiplication with $L^\infty(m)$.

$$L^\infty(m) \times M \to M.$$

$$f(gv) = (fg)v.$$

$$1v = v.$$

$$\forall f, g \in L^\infty(m), v \in M.$$

(II) ptwise L^2 ~~(m)~~ ~~norm~~:

$$|fv| \leq |f|\,|v|.$$

$$\|v\|_M = \sqrt{\int_M |v|^2 \, dm}.$$

Basic example: $L^2(V)$ for V a normed bundle.

Smooth: describe v.b. via fibres or C^∞ sections.

non-smooth: latter remains; ie via sections. No REAL FIBRES.

Basic feature: locality $\forall v, w \in M$, $E \subset X$ Borel.

$v = w$ ~~if~~ m-a.e. on E.

provided $\chi_E(v - w) = 0 \iff |v - w| = 0$, m-a.e.

Dual of M, M^*. space of lin cts maps.

$L : M \to L^1(m)$. which are local.

$$L(fv) = f L(v) \quad \forall v \in M,\ f \in L^\infty(m).$$

M^* L^2-normed L^∞-module

$$|L|_* := \operatorname*{ess\text{-}sup}_{|v| \le 1 \ m\text{-a.e.}} |Lv|.$$

Dual of $L^2(m)$: $\quad L(f) = \int fg \, dm$;

Dual of $L^2(m)$ as a <u>module</u> is $L^2(m)$.

$$T : L^2(m) \to L^1(m).$$

Ie, $\exists!\, g \in L^2(m)$. $T(f) = fg$ m-a.e. $\forall f \in L^2(m)$.

<u>Recall Sobolev function S^2.</u>

$$\left.\begin{array}{l} (f_n) \subset S^2(X). \\ f_n \to f \quad m\text{-a.e.} \\ |Df_n| \rightharpoonup G \text{ in } L^2(m). \end{array}\right\} \Rightarrow \quad \begin{array}{l} f \in S^2(X) \\ G = Df. \end{array}$$

+ Rules for diff.

<u>Pre-Cotangent-Module</u>

$$PCM = \left\{ (A_i, f_i)_{i \in \mathbb{N}} . \quad \begin{array}{l} A_i \text{ Borel partition.} \\ f_i \in S^2(X) \quad \forall i \in \mathbb{N}. \\ \sum_i \int_{A_i} |Df_i|^2 \, dm < \infty \end{array} \right\}.$$

$\sim$ on PCM:

$(A_i, f_i)_{i\in\mathbb{N}}$ $(B_i, g_i)_{i\in\mathbb{N}}$ s.t.

$\forall i,j$ $\quad |D(f_i - g_j)| = 0$ m-a.e. on $\{A_i \cap B_j\}$.

$[A_i, f_i]$ = equiv class of $(A_i, f_i)_{i\in\mathbb{N}}$.

$PCM/\sim \ni [A_i, f_i] \longmapsto \sum_i \chi_{A_i} df_i$ [In smooth setting]

Similarly,

the functions $h[A_i, f_i] := [A_i \cap E_j, \alpha_j f_i]$ $\quad h = \sum_j \alpha_j E_j$.

$|[A_i, f_i]| := |Df_i|$.

$\|[A_i, f_i]\| = \sum \sqrt{|Df_i|^2}$

done in smooth setting, so, take this as inspiration, and so, write

$L^2(T^*X)$ as the completion of $(PCM/\sim, \|\cdot\|)$.

Its elements are called 1-forms.

$f \in S^2(X)$, $\quad df \in L^2(T^*X)$ via

$$df := [X, f].$$

Then $|df| = |Df|$ m-a.e.

If $X = M \in C^\infty$ mfld, $L^2(T^*X)$ identified as the space of L^2-sections.

$d : S^2(X) \to L^2(T^*X)$ is a densely-defined, closed operator.

<u>Def^{n}</u>. Tangent module $L^2(TX) = L^2(T^*X)^*$, elements vector fields.

<u>Rmk</u> In many situations <u>reflexive</u>!

Sps $X \in L^2(TX)$ satisfies:

~~$L^2(TX) \ni$ grad~~ $df(X) = \frac{1}{2}|df| + \frac{1}{2}|X|$,

then write grad $f = X$.

$$\begin{tikzcd}
S^2(X) \arrow[rr, "d"] \arrow[dr, "\text{Derivation}"'] & & L^2(T^*X) \arrow[dl, "\text{v. field}"] \\
& L^1(m) &
\end{tikzcd}$$

diagram commutes.

<u>Deformations</u> $\varphi : X_1 \to X_2$ Lipschitz.

$\mathrm{Lip}(\varphi) < \infty$, $\varphi_* m_1 \le C m_2$. $\exists C > 0$.

Say <u>bounded deformations</u>.

Why such a notion: $f \in S^2(X_2)$, $\Rightarrow f \circ \varphi \in S^2(X_1)$.

This allows pullback of 1-forms. Such a unique map exists!.

Given mm. spaces; TFAE:

1) $W^{1,2}$ Hilbert.
2) $\forall X, Y \in L^2(TX)$, have then that $|\cdot|$ polarises.

↑ gives $\langle X, Y \rangle \in L^1(m)$. ← might want to call the metric tensor.

<u>3 simple formulas Slide.</u> ① + ② enough to define Covariant derivative <u>without</u> Lie bracket.

Also extend d. to higher forms → $d^2 = 0$!

<u>Def^{n}s</u> invariant for mutually abs cts measures!

On $RCD(k,\infty)$ $\exists$ weak Bochner.

$$\Delta \frac{|\nabla f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + k|\nabla f|^2.$$

Bakry understood '85 by purely algebraic methods insert $|Hf|^2_{HS}$ on RHS.

$$\int |Hf|^2_{HS}\, dm \leq \int k|\nabla f|^2 + |\Delta f|^2.$$

$\Downarrow$

$W^{2,2}(X)$ defined.

Def^n of $W^{1,2}_c(TX)$ of covariant derivative, which is compatible and torsion free.

+ $W^{1,2}_d(T^*X) \Longrightarrow$ de Rham Cohomology & Hodge.

Ricci Curvature!

$$\mathrm{Ric}(X,X) := \Delta \frac{|X|^2}{2} - |\nabla X|^2_{HS} + \langle X, \Delta_H X \rangle.$$

which satisfies $\mathrm{Ric}(X,X) \geq k|X|^2 m$.

measure-valued operator satisfying.

"Spaces with Ricci curvature bounded from below have Ricci curvature bounded from below."