N. Gigli   Lecture 4   26/02/2014.

- The splitting theorem: (M,g) C^∞, complete, $Ric \geq 0$, contains a line. Then $M = N \times \mathbb{R}$, N Riem.

→ In M^2, where all curvatures are the same, $M^2 = \mathbb{R}\times\mathbb{R}$ or $\mathbb{R}\times S^1$, because $Ric_M \geq 0$ means it doesn't have saddle points.

- Almost splitting: (M,g) Riem, $Ric \geq -\varepsilon$ contains ~~g~~ geodesic of length L, $\varepsilon, L^{-1} \ll 1$, then a big portion of M is mGH close to product.

↓

Can apply this to spaces with $Ric \geq 4g$, because, by scaling the manifold up, say, by a large factor, tends $Ric_{kM} \geq \varepsilon_k g$. $\varepsilon_k \to 0$ as $k \to \infty$.
k-scale

Then apply ~~splitting~~, almost splitting. Almost splitting says something about local structure of the space!

Corollary: Splitting for limit spaces: (X,d,m) pmGH limit of uniform dim bound. → M_n with $Ric(M_n) \geq -\varepsilon_n$, $\varepsilon_n \to 0$, X contains a line $\Rightarrow$ splits off a factor $\mathbb{R}$.

pmGH convergence ⇝ So M_n must contain long geodesics b/c X contains a line. pointed mGH

Thm (G. '13) $(X,d,m) \in RCD^*(0,N)$ containing a line.
Then $\exists\ (X',d',m')$ such that

$$(X,d,m) \ \text{isometric} \ \text{to} \ (X'\times\mathbb{R},\ d'\otimes d_{Eucl},\ m'\times\mathcal{L}^1).$$

$$(d'\otimes d_{Euc})((x',t),(y',s))^2 = d'(x',y')^2 + |t-s|^2.$$

(I) $N \geq 2$, then (X',d',m') is $RCD^*(0,N-1)$.
(II) $N\in[1,2)$, X' contains only a point.

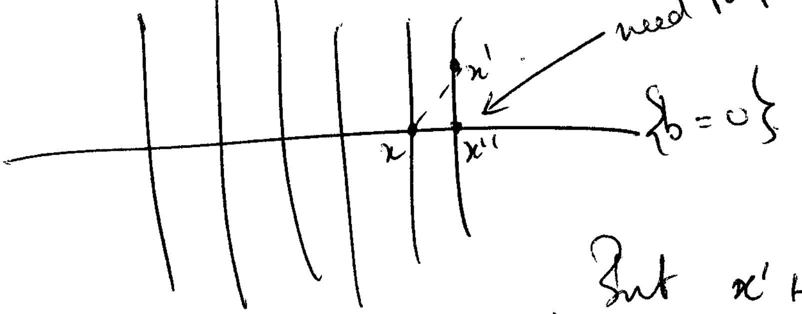
Pf. Via Busemann functions. But no Bochner identity or Hessian, so only inequality, and

$$\Delta \nabla b^+ \cdot \nabla f = \nabla b^+ \cdot \Delta \nabla f.$$

and $T: X_1 \to T_2$ Borel, a.e. invertible,

$$\| f\circ T\|_{W^{1,2}(X_1)} = \|f\|_{W^{1,2}(X_2)}.$$

$\nearrow$ This is only valid b/c we're in RCD^*. In general metric space, $W^{1,2}$ is not rich enough to encode ~~full~~ sufficient geometric information about the space.



minimum $d(x,x')$ ~~obtained~~ achieved at x''. on line.

But $x' \mapsto d^2(x,x')$ is only Lipschitz, and need at least C^1 to use ~~Euler~~ PDE.

Instead, $t \mapsto \int f\, d\mu_t$ for $f\in W^{1,2}$ is C^1 and run argument in W_2 Wasserstein.

space. I.e.,

$$\int b^+ \, dF_\#(\psi) \, d\mu = \int b^+ \, d\nu .$$

for all measures ν, μ and F transport? (see sketch).

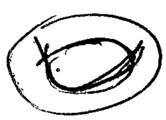
But we can let $\mu \to \delta_x$, $\nu \mapsto \delta_y$ etc, to obtain information ~~downstairs~~ downstairs.

Thm. (Ketterer '13). (X, d_X, m_X) m.m. space, (Y, d_Y, m_Y) N-cone. TFAE.

$\to$ $(\cancel{X} d_X, m_X) \in RCD^*(N-1, N)$.

$\to$ $(Y, d_Y, m_Y) \in RCD^*(0, N+1)$.

Rmk ~~In general~~ In general, conical singularity is topological not geometric. I.e., if the fibres we had

I.e., $M =$  torus, (not allowed in prev)

Thm. Then conical point is very bad.

Diff to $M = S^n$. $\leftarrow$ can write cone as graph of Lip function, So can move top. singularity to metric.

Rmk. Typically conical spaces are not limits of RCD! Obviously $M = S^n$ is, by smoothing ~~[illegible]~~ near conical point.