Nicola Gigli   Lecture 3   25/02/2015.

$W^{1,2}$ Hilbert $\Leftrightarrow$ L^2 grad flow linear.

$\Updownarrow$ $\Leftrightarrow$

$E(f) = \frac{1}{2}\int_X |Df|^2 \, dm$ satisfies polarization identity.

$\Updownarrow$

$\partial^- E(f)$ is linear in f.

- Heat flow as EVI$_K$ gradient flow of the entropy.

$t \longmapsto \rho_t m =: \mu_t$.

Show $t \longmapsto \frac{1}{2} W_2^2(\mu_t, \nu)$ - grad flow $\longleftarrow$ Easy

$t \longmapsto \frac{1}{2} \mathrm{Ent}_m(\mu_t)$ grad flow $\longleftarrow$ tricky!

Thm Rajala '12: X, compact $CD(K,\infty)$ $\mu, \nu \in \mathcal{P}(X)$.
s.t. $\mu, \nu \le Cm$. Then $\exists \mu_t$ s.t. $\mu_t \le C'm$ $\forall t \in [0,1]$
and $t \longmapsto \mathrm{Ent}_m(\mu_t)$ K-convex.

Thm (Metric Brenier's thm) (Ambrosio, G., Savaré).
μ_t geodesic, $\mu_t \le Cm$ $\forall t \in [0,1]$, $\pi \in \mathcal{P}(C([0,1], X))$
a lifting of μ_t and φ Kantorovich pot. Then π
represents the gradient of $-\varphi$.

~~$s \longmapsto \nu_s$ geodesic~~
Derivative of $\mathrm{Ent}_m(\nu_s)$.

$s \longmapsto \nu_s$ geodesic from Rajala's thm, φ Kantorovich pot.

Show . $\liminf_{s \downarrow 0} \frac{\mathrm{Ent}_m(\nu_s) - \mathrm{Ent}_m(\nu_0)}{s} \geq - \int \nabla \eta \cdot \nabla \varphi \, dm.$

$= - \int \eta \, \Delta \varphi \, dm.$

A.G.S. "11": $\frac{d}{dt} \frac{1}{2} W_2^2(\mu_t, \nu) \leq \frac{d}{ds}\Big|_{s=0} \mathrm{Ent}_m(\nu_{t,s}).$

$\leq \mathrm{Ent}_m(\nu) - \mathrm{Ent}_m(\mu_t) - \frac{k}{2} W_2^2(\mu_t, \nu)$

and also, contractivity: $(\mu_t), (\nu_t) \subset \mathcal{P}(X)$ heat flows,

$W_2^2(\mu_t, \nu_t) \leq e^{-2kt} W_2^2(\mu_0, \nu_0).$

So, we get heat kernel + Brownian motion.

If nonlinear, meaningless b/c you can't say anything about sum of heat flows $\mu_t + \nu_t$.

Bochner inequality

Smooth case: $\Delta \frac{|\nabla f|^2}{2} \geq \nabla f \cdot \nabla \Delta f + k |\nabla f|^2 \left(+ \frac{(\Delta f)^2}{N} \right)$

but we're on $N = \infty$ so this term 0.

Thm (Kuwada '09.)

Let $H_t : \mathcal{P}(X) \to \mathcal{P}(X)$ heat flow on measures and $h_t : L^1 \to L^1$ for densities. Then TFAE:

$W_2^2(H_t(\mu), H_t(\nu)) \leq e^{-2kt} W_2^2(\mu, \nu) \qquad \forall t \geq 0, \mu, \nu \in \mathcal{P}(X).$

$\mathrm{lip}^2(h_t(f)) \leq e^{-2kt} h_t(\mathrm{lip}^2(f)) \qquad \forall t \geq 0, f: X \to \mathbb{R}$ Lipschitz.

$\mathrm{lip}(f)(x) := \liminf_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}$

- ~~Lip~~ $\mathrm{Lip}(X) \cap S^2$ dense in $W^{1,2}$, so, we get $|D h_t(f)|^2 \le e^{-2Kt} h_t(|Df|^2)$. $\forall t \ge 0$, $f \in W^{1,2}(X)$.

and this gives

$$\int \Delta g \frac{|Df|^2}{2} \, dm \ge \int (\nabla f \cdot \nabla \Delta f + K |Df|^2) g \, dm.$$

$\forall f \in W^{1,2}(X) \cap \mathcal{D}(\Delta)$, $\Delta f \in W^{1,2}(X)$, $g \in L^\infty(X) \cap \mathcal{D}(\Delta)$

Also converse from Bochner to $RCD(K,\infty)$.

For $N < \infty$, on $RCD^*(K,N)$ (define this as follows):

$$\int \Delta g \frac{|Df|^2}{2} \, dm \ge \int \left(\frac{(\Delta f)^2}{N} + \nabla f \cdot \nabla \Delta f + K |Df|^2 \right) g \, dm$$

- Get Li-Yau + Lipschitz regularity for Harmonic maps.

Optimal maps: $(X,d,m) \in RCD^*(K,N)$, $\mu, \nu \in \mathcal{P}(X)$, $\mu \ll m$.

- $\exists$ only one optimal plan.
- plan induced by T.
- μ-a.e. x, $\exists$ only a single geodesic γ^x from x to $T(x)$
- μ-a.e. $x \ne y$.

Distributional Laplacian

For just RCD, assume (X,d,m) infinitesimally Hilbertian, locally compact, $\Omega \subset X$ open, $g \in S^2(\Omega)$.

$g \in \mathcal{D}(\Delta, \Omega)$ $\exists$ Radon measure μ.

$$\int \Delta g \, dm = \cancel{\int g \, d\mu} \qquad \Delta g = \mu.$$

• Grad. linearity, product, chain rule.

Thm (G. Mondino.)

(X, d, m) inf. Hil. + doubling + 2-Poincaré.

$\Omega \subset X$, $g \in S^2(\Omega)$.

TFAE.

- $g \in D(\Delta, \Omega)$ $\Delta g \le 0$.
- $\forall f \in \mathrm{Lip}(X)$, $f \ge 0$, $\mathrm{spt}\, f \subset \Omega$.

$$\int_\Omega |Dg|^2 \, dm \le \int_\Omega |D(g+f)|^2 \, dm.$$

Def^n of subsolutions.

Euler–Lagrange in non-smooth.

Laplace Comparison

on Riem. M. $\mathrm{Ric} \ge 0$, $\dim \le N$.

$$\Delta \tfrac{1}{2} d^2(o, x) \le N$$

in distributional sense.

Same holds for $RCD^*(0,N)$.

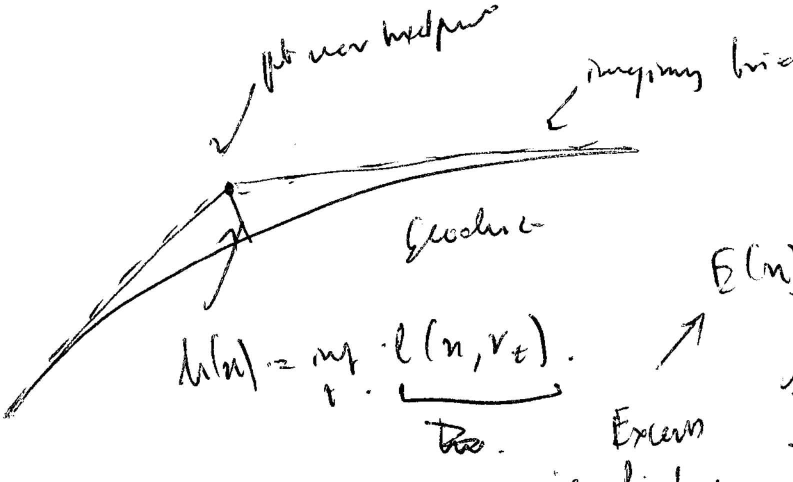
Abresch–Gromoll fails in $CD(K,N)$ in general.

Pf of inequality relies on.

- Laplace comparison for distance
- linearity of Lap.
- Weak max principle.

Satisfied in $RCD(K,N)$.

• Abresch–Gromoll inequality.



on Riem mfld:

$E(h) \le f_{K,N}(h/h)$.

Excess, i.e, failure of triangle inequality.

$$\frac{f_{K,N}(h)}{h} \to 0 \text{ as } h \to 0.$$

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