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· Variational def^n of $|Df|$

$$|f(\gamma_1)-f(\gamma_0)| \le \int_0^1 G(\gamma_t)\,|\dot\gamma_t|\,dt. \quad \gamma_t\in C^\infty,\ f\in C^\infty,$$

$|Df|$ is the minimum G

· This def^n is not good for Sobolev functions, b/c things can be bad on a set of measure 0.

$\pi\in\mathscr{P}(C([0,1],X))$, test plan if. $\exists C>0$,

(I). $e_{t\#}\pi \le C m$. $\forall t\in[0,1]$. $e_t\in C(C([0,1],X),)\to X$, $\gamma\mapsto\gamma_t$.

(II).

Then $f: X\to\mathbb{R}$ belongs to $S^2(X,d,m)$] Sobolev ~~space~~ & class.

$$\int|f(\gamma_1)-f(\gamma_0)|\,d\pi(\gamma) \le \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,dt\,d\pi(\gamma).$$

$\forall$ test plan π. $\leftarrow$ iff for Sobolev on $\mathbb{R}^n$.

Then a minimal G exists and denote it finally by $|Df|$.

Can define energy $E: L^2(X,m)\to[0,+\infty]$,

$$E(f) := \frac{1}{2}\int|Df|^2\,dm \quad \forall f\in S^2(X), \quad +\infty \text{ otherwise}.$$

E convex, lower semicontinuous

(Can also define $E(f) = \inf\limits_{\substack{f_n\xrightarrow{L^2}f\\ f_n\ \mathrm{Lip}}} \liminf\limits_{n\to\infty} \frac{1}{2}\int|\mathrm{lip}\, f_n|^2\,dm$.)

<u>Rmk</u>. Nothing guarantees this is quadratic form. Df is not a real object, if it were, polarise.

Sobolev space. $W^{1,2}(X) = L^2(X) \cap S^2(X)$.

with. $\|f\|_{W^{1,2}(X)} = \|f\|_{L^2} + \| |Df| \|_{L^2}$.

<u>Laplacian</u> $f \in \mathcal{D}(\Delta) \subset W^{1,2}(X)$ if $\partial^- E(f) \neq 0$.

Define $\Delta f := -v$ v is the minimal norm in $\partial^- E(f)$.

* This is a Th^m on a Riem mfld.

"Integration by parts": $f \in \mathcal{D}(\Delta)$ $g \in W^{1,2}(X)$,

$$\left| \int g \Delta f \, dm \right| \leq \int |Dg| |Df| \, dm.$$

<u>Pf</u> 1-line: $\int (\Delta f)(\varepsilon g) \leq E(f + \varepsilon g) - E(f)$. (semicontinuity)

$$\leq \frac{1}{2} \int (|Df| + |\varepsilon| |Dg|)^2 - |Df|^2$$

$$= |\varepsilon| \int |Df| |Dg| + o(\varepsilon).$$

$\boxed{\Delta \text{ linear iff } W^{1,2}(X) \text{ Hilbert}}$

$\forall f_0 \in L^2(X,m)$ $\exists$ unique $t \mapsto f_t \in L^2(X,m)$ s.t.

$$\frac{d}{dt} f_t = \Delta f_t. \qquad \forall t > 0$$

<u>Th^m</u> (G., Kuwada, Ohta, Ambrosio, G., Savaré '11).

(X,d,m) $CD(K,\infty)$ $\mu = fm \in \mathcal{P}_2(X)$. $f \in L^2(X,m)$.

$t \mapsto f_t$ GF. w.r.t. L^2 starting from f.

$t \mapsto \mu_t$ GF. of Ent_m w.r.t. W_2. starting from μ.

Then $\mu_t = f_t m$.

<u>Details in slides!</u>

<u>Def^n</u> $RCD(K,N) := CD(K,N) +$ linearity of heat flow.

$= CD(K,N) + W^{1,2}$ is Hilbert.

Goal: Develop a refined enough calculus to detect when $W^{1,2}$ Hilbert.

Forget:	Focus:	Because
Lipschitz functions.	Sobolev functions.	RCD defined via Sobolev.
Charts.	intrinsic calculus	when do charts exist?
Def. of Df via ∇f.	understand duality $Df(\nabla g)$.	

* Note about charts: doing calculus through charts → is it that the charts are less than smooth or the space? Better to do intrinsic.

Gradients on $\mathbb{R}^d$:

$$Df(x)(w) \leq \|Df(x)\|_* \|w\| \leq \frac{1}{2}\|Df(x)\|_*^2 + \frac{1}{2}\|w\|^2.$$

w vector: can say $v = \nabla f(x)$ iff equivalently,

$$Df(x)(v) \geq \frac{1}{2}\|Df(x)\|_*^2 + \frac{1}{2}\|v\|^2.$$

Rmk: uniqueness iff norm is strictly convex. Linearity iff norm arises from inner product.

Important identities:

$$\max_{v \in \nabla g(x)} Df(v) = \inf_{\varepsilon>0} \frac{\|D(g+\varepsilon f)\|_*^2(x) - |Dg|_*^2(x)}{2\varepsilon}.$$

$$\min_{v \in \nabla g(x)} Df(v) = \sup_{\varepsilon>0} \quad \cancel{|D(g+\varepsilon} \quad (\cancel{\ }\ ).$$

Thus, we can define Df via this on X:

$$D^+f(\nabla g) := \inf_{\varepsilon>0} \frac{|D(g+\varepsilon f)|^2 - |Dg|^2}{2\varepsilon}.$$

$$D^-f(\nabla g) := \sup_{\varepsilon>0}$$

Locality: $D^{\pm}f(\nabla g) = D^{\pm}\tilde{f}(\nabla\tilde{g})$, m-a.e. in $\{f=\tilde{f}\}\cap\{g=\tilde{g}\}$.

Chain rule + Leibnitz rule – check slides.

Def^{n} G'll: (X, d, m) infinitesimally Hilbertian if $W^{1,2}$ Hilbert

In this case:

$$D^{+}f(\nabla g) = D^{-}f(\nabla g) = D^{+}g(\nabla f) = D^{-}g(\nabla f), \quad m\text{-a.e.}$$

Define $\nabla f \cdot \nabla g$ as this. ↖ Abstract version of Riesz-Rep!

$\forall g \in S^2$, $\pi \in \mathcal{P}(C([0,1], X))$

$$\limsup_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma_0)}{t}\, d\pi \le \frac{1}{2}\int |Dg|^2(\gamma_0)\, d\pi + \limsup_{t\downarrow 0} \frac{1}{2t}\iint_0^t |\dot{\gamma}_s|^2\, ds\, d\pi$$

If opposite inequality holds. Say π represents ∇g.

(Exactly as before when we asked $v = \nabla g$, but now this is in $\mathcal{P}(\text{—})$.)

In C^∞ world: $T(x)_t = \exp_x(t\nabla g(x))$, then

$\pi := T_* \mu$. ← not unique, there is no unique way of going in a certain direction.

First order diff formula: $f, g \in S^2$, π representing ∇g.

$$\int D^{+}f(\nabla g)(\gamma_0)\, d\pi \ge \limsup_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\, d\pi$$

$$\ge \liminf_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\, d\pi \quad \leftarrow \text{horizontal perturb.}$$

$$\ge \int D^{-}f(\nabla g)(\gamma_0)\, d\pi. \quad \leftarrow \text{vertical perturb.}$$

Prop of k-convex function on $\mathbb{R}^d$: E k-convex, $t \mapsto x_t$.

s.t. $x_t' = -\nabla E(x_t)$.

and $y_{t,s} = (1-s)x_t + sy$,

$$\frac{d}{ds}\Big|_{s=0} E(y_{t,s}) = \nabla E(x_t)(y - x_t).$$

and hence

$$\frac{d}{dt}\frac{1}{2}|x_t - y|^2 \le E(y) - E(x_t) - \frac{k}{2}|x_t - y|^2.$$

So on X, replace $|x_t - y|$ by $d(x_t, y)$, to define $(x_t) \subset (Y, d_Y)$ sm. EVI_k-GF of a $E: Y \to [0, +\infty]$ loc. ab. ct. and $\forall y$ this exp. is satisfied.

Savaré shown that if (x_t) is EVI_k,

$$E(x_0) = E(x_t) + \frac{1}{2}\int_0^t |x_s'|^2 + |\partial E|^2(x_s)\, ds \quad \forall t > 0.$$

Vice versa not true.