Nicola Gigli — Spaces with Ricci curv. bdd from below — 23/02/2015.

- Not much for upper Ricci bd. → does not give useful geometric information. I.e., $\exists$ neg. Ricci metric on S^3.

Sectional curv. } → Topology = GH convergence → only concerns the metric!.

Synthetic notion: Alexandrov Spaces.

Ricci bdd below } → GH convergence does not work.

$\exists (M_i, g_i)$ smooth → (M, g) smooth but Ricci is not preserved.

Need: measured GH convergence, and metric alone is not enough.

Synthetic notion: $\underbrace{CD(K,N)}_{\text{Breakthrough by Villani}} / \underbrace{RCD(K,N)}_{\text{later}}$.

- $Ric \geq Kg \iff$ relative entropy K-convex.

$\mathrm{Ent}_{\mathrm{vol}}(\mu) \int \rho \log \rho \, d\mathrm{vol}$, $\mu = \rho \, d\mathrm{vol}$.

Take ($\Leftarrow$) as def^{n} for metric spaces.

$CD(K,\infty)$ - metric measure spaces $Ric \geq Kg$. via K-convex.
def^n

Finsler structures are included!

<u>Diff:</u>

Analysis:	Geometry.
Tangent/cotangents cannot be identified.	no Abresch-Gromoll ineq.
No unif. Dirichlet fun.	no splitting Th^m.

Even with lower bound on Ricci, Finsler can be very diff.

But, if $(M_i, g_i) \longrightarrow (M, \mu) \longleftarrow$ <u>cannot</u> be Finsler,

So want to find subset of CD so that limits are <u>not</u> Finsler.

<u>M Finsler, TFAE:</u>

- M Riemannian.
- $W^{1,2}$ Hilbert.
- Heat flow linear.

<u>Heat Flow:</u>

- Gradient flow of Dirichlet energy w.r.t. L^2
- Grad flow of rel. entropy w.r.t. W_2.

The idea: Restric $CD(K,\infty)$ so that $W^{1,2}$ Hilbert, because Riem computations on smooth manifolds should carry!

Difficulty: Show resulting notion is stable w.r.t. mGH.

~~But~~ mGH, GH are 0^{th} order conditions, but $W^{1,2}$ condition is 1^{st} order. So need more.

Need to use: • $W^{1,2}$ Hilbert $\Leftrightarrow$ L^2 grad flow of Dirichlet is linear.

• L^2 grad flow of Dirichlet energy coincides with W_2-grad flow of rel. entropy

• W_2-gradient flow of rel. entropy is stable.

Grad Flows

$$|\dot{x}_t| := \lim_{h\to 0} \frac{d(x_{t+h}, x_t)}{h} \qquad x_t \text{ abs cts curve.}$$

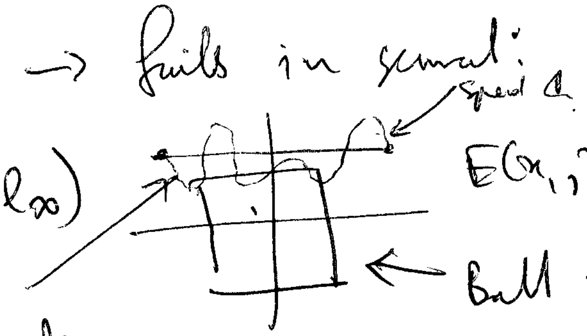
$$|\partial^- F|(x) := \overline{\lim_{y\to x}} \frac{(F(x)-F(y))^+}{d(x,y)}.$$

$$F(x_0) \leq F(x_t) + \frac{1}{2}\int_0^t |\dot{x}_s|^2 + |\partial^- F|^2(x_s)\, ds \quad \forall t>0$$

($\mathbb{R}^n$, $|\partial^- f| = |\nabla F|$ and equality holds iff $\dot{x}_t = \nabla F(x_t)$.
So use that as motivation!)

Existence: Space compact + $F(x_0) < \infty$ (A., G., S. '04).

Uniqueness $\to$ fails in general:

$(\mathbb{R}^2, l_\infty)$ 

$E(x_1, x_2) = x_1$.

but also has speed ≤ 1 and metric doesn't see it.

Th^m G. '09 (X, d, m) compact, $CD(k, \infty)$.

$\forall \mu \in \mathbb{P}_2(X)$, $\mathrm{Ent}_m(\mu) < \infty$, GF of Ent_m starting from μ exists and unique

This is stable under mGH convergence of base space.

Def^n $(X_n, d_n, m_n) \xrightarrow{mGH} (X_\infty, d_\infty, m_\infty)$ if $\exists$ isom embeddings

$$i_n, i_\infty : X_i, X_\infty \longrightarrow \underbrace{(Y, d_Y)}_{\exists \text{ Big space}} \quad \text{s.t.}$$

$$(i_n)_\# m_n \longrightarrow (i_\infty)_\# m_\infty \quad (\text{weakly}).$$

No distance! Check away distance. Only opt of measure is preserved

Say $n \mapsto \mu_n \in \mathcal{P}_2(X_n)$ converges to $\mu_\infty \in \mathcal{P}_2(X_\infty)$.

if $\quad (i_n)_\# \mu_n \longrightarrow (i_\infty)_\# \mu_\infty$.

$\underline{\text{Th}^{m}}$ (Lott - Sturm - Villani).

Ex. • $\mathrm{Ent}_{m_\infty}(\mu_\infty) \le \liminf_{n\to\infty} \mathrm{Ent}_{m_n}(\mu_n)$.

$\forall$ sequences μ_n weakly converge to μ_∞.

• $\forall \mu_\infty$ $\exists$ μ_n weakly conv to μ_∞ s.t.

$\mathrm{Ent}_{m_\infty}(\mu_\infty) \geq \limsup_{n\to} \mathrm{Ent}_{m_n}(\mu_n)$.

$\underline{\text{Corollary}}$ $CD_?(k,\infty)$ stable under mGH.

$\underline{\text{Th}^{m}}$ $\underline{(G\ '09)}$: $(\partial^- \mathrm{Ent}_{m_\infty})(\mu_\infty) \le \liminf_{n\to\infty} |\partial^- \mathrm{Ent}_{m_n}|(\mu_n)$.

when $X_n \xrightarrow{mGH} X_\infty$.

$\underline{\text{Corollary 1}}$ $\lim_{n\to\infty} \mathrm{Ent}_{m_n}(\mu_n) - \mathrm{Ent}_{m_\infty}(\mu_\infty) < \infty$.

Then GF of Ent_{m_n} startin from μ_n $\longrightarrow$.

GF of Ent_{m_∞} startin from μ_∞.

$\underline{\text{Corollary 2}}$ '$CD(k,\infty)$ + linearity of the GF. of entropy'

is closed w.r.t. mGH.