Super Ricci flow — Sturm. 02/04/2015.

(X, d_t, m_t), $m_t = e^{-f_t} m$, $m(X) = 1$, $|f_t(x)| \leq C$.

$$\frac{d_t(x,y)}{d_s(x,y)} \leq e^{C|t-s|}. \qquad \forall x, y, s, t.$$

$\Updownarrow$

All this is true for Wasserstein distance W_t for d_t.

Ie,
$$\frac{W_t(\mu,\nu)}{W_s(\mu,\nu)} \leq e^{C^*|t-s|} \qquad \forall \mu, \nu, s, t.$$

Pf. $\mu = \delta_x$, $\nu = \delta_y$, Choose q_s opt. coupling for d_s.

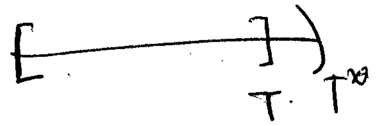
$$\begin{aligned} W_t^2(\mu,\nu) &\leq \int d_t^2(x,y)\, dq_s(x,y) \\ &\leq e^{2C^*|t-s|} \int d_s^2(x,y)\, dq_s(x,y) \\ &= W_s^2(\mu,\nu). \end{aligned}$$

Classic Riem. picture: $\Updownarrow$ $\dfrac{g_t}{g_s} \leq e^{2C^*|t-s|}$.

(M, g_t) evolving $\Leftrightarrow |\partial_t g_t| \leq 2C^* g_t$.

under Ricci flow. $\Leftrightarrow |Ric_{g_t}| \leq C^* g_t$. on $[0,T] \times M$.

away from singularities



Wasserstein: $|\partial_t W_t| \leq 2C^* W_t$.

Now consider $S : [0,T] \times \mathcal{P}_2 \to (-\infty, \infty]$,

$$(t,\mu) \mapsto Ent(\mu | m_t) = Ent(\mu, m) + \int_X f_t \, d\mu.$$

$$\mathcal{A}_t(\mu) = \lim_{k\to\infty} 2^k \sum_{i=1}^{2^k} W_t^2\left(\mu^{i2^{-k}}, \mu^{(i-1)2^{-k}}\right) = \int_0^1 G_t^a(\mu)\, da.$$

for Lipschitz curve $\mu : [0,1] \to \mathcal{P}$. inf action = (met. dist.)^2.

$$\mathcal{H}_t(\mu) = \lim_{k\to\infty} 2^k \sum_{i=1}^{2^k} \partial_t^- W_t^2\left(\mu^{i2^{-k}}, \mu^{(i-1)2^{-k}}\right) = \int_0^1 \mathcal{H}_t(\mu)\, dt.$$

$$\partial_t^- u(t) = \liminf_{s\nearrow t} [u(t) - u(s)]$$

<u>Def</u>. (X, d_t, m_t) is a super Ricci flow. $\Leftrightarrow$

(I) S is backward dyn. convex on $(\mathcal{P}_2, W_2)$

(II). super-N-Ricci flow $\Leftrightarrow$ backward dynamically N-convex.

$\forall \mu^0, \mu^1 \in \mathcal{P}$, $\forall t \in [0,T]$ s.t. $S_t(\mu^0) < \infty$. $S_t(\mu^1) < \infty$

$\exists\, W_t^*$-geodesic μ^a. $\forall a \leq a^*$.

$$\frac{1}{a}\left[\Phi_N(S_t(\mu^0) - S_t(\mu^a)) + \Phi_N\left(S_t(\mu^1) - S_t(\mu^{1-a})\right)\right]$$
$$\geq -\frac{1}{2}\partial_t^- W_t^2(\mu^0,\mu^1) - C^* a W_t(\mu^0,\mu^1) + \frac{1}{N}[S_t(\mu^0) - S_t(\mu^1)]$$

(III). a.e. Super Ricci flow $\nearrow$ a.e. ...

$$\int_r^s \Phi_N(S_t(\mu^0) - S_t(\mu^a)) + \Phi_N(S_t(\mu^1) - S_t(\mu^{1-a}))\, dt.$$
$$\geq -\frac{1}{2}\left[W_s^2(\mu^0, \mu^1) - W_r^2(\mu^0,\mu^1)\right] - C^* a \int_r^s W_t^2(\mu^0,\mu^1)\, dt$$
$$+ \frac{1}{N}\int_r^s [S_t(\mu^0) - S_t(\mu^1)]^2\, dt.$$

Convergence of time dep mms. $\quad I = [0,T]$.

$$\mathbb{D}_I\left((X, d_t, f_t, m), (X', d'_t, f'_t, m') \right)_{t\in I}$$

$$= \inf \Big\{ \iiint_{X X' T} \hat{d}_t^2(x,y)\, dt\, d\hat{m}(x,y)$$

$$+ \iiint (f_t(x) - f'_t(y))\, dt\, d\hat{m}(x,y) :$$

$$\hat{m} \in \mathrm{Cpl}(m, m'),\ \hat{d} \in \mathrm{Cpl}(d, d') \Big\}.$$

$\mathbb{X}_N = \{$ time dep mms $(X, d_t, f_t, m)_{t\in[0,T]}$,
all the above bounds with C and
Lipschitz bounds with C^*, δ, $\dim \le C$,
$x \mapsto f_t(x)$ lsc, N-super-Ricci-flows $\}$.

Thm $\mathbb{X}_N$ is closed w.r.t. $\mathbb{D}_I$ $\forall N \in [1,\infty]$.

Thm $\mathbb{X}_N^{*,\delta} = \{$ time dependent mms with bounds C, C^*, $\dim \le C$,
$(t,x) \mapsto f_t(x)$; C-Lip $\}$

is $\mathbb{D}_I$-compact.

<u>Summary</u> Class of time-dep mms with $\forall t$.

$\forall \mu^0, \mu^1 \quad \rightsquigarrow \quad -\partial_a S_t(\mu^0) + \partial_a S_t(\mu^1)$

$$\geq -\frac{1}{2}\partial^- W_t^2(\mu^0,\mu^1) + \frac{t}{N}\left(W_t^2(\mu^0,\mu^1)\right)$$

is <u>compact</u>. $\boxed{\text{To, } \partial W_t \geq \_\_}$

Super Ricci flow.

Get Ricci flow (hope). is to choose minimal, So there is $= ?$

From dyn. Convexity to EVI etc.

(X, d_t), S dyn. Convex, $(x_t)_{t \geq t_0}$ Smooth.

$\updownarrow$

(M, g_t) smooth. On Riem mfld. $\mathrm{Hess}_t\, S_t \geq \frac{1}{2}\dot{g}_t$.

<u>Prop.</u> <u>TFAE</u>.

(I) $\dot{x}_t = -\nabla_t S_t(x_t)$.

(II) $\frac{1}{2}\partial_s d_t^2(x_s, z)\big|_{s=t} + \frac{1}{2}H_t^0(x_t) \leq S_t(z) - S_t(x_t)$.

$\forall$ geo γ x_t to z, here:

$$H_t^0(x_t) = \int_0^1 (1-b)\, h_t^b(\gamma)\, db$$
$$= \int_0^1 (1-b)\, \dot{g}_t^b(\gamma)\, db.$$

Consequences of Prop (II):

(III) $\cdot\ \partial_t\, d_t^2(x_t, y_t) \leq 0$. $\left(\leq -\frac{1}{N} \cdot |S_t(x_t) - S_t(y_t)| \right.$ for N-S.R.F $\Big)$.

(IV) Dynamic convexity: