Dynamic Convexity   Sturm.   01/04/2015

Frame $(X, d_t)_{t\in[0,T]}$, d_t geodesic, complete metric, seperable.

$$C^{-1} \le \frac{d_s}{d_t} \le C \quad \forall s,t.$$

Concept of Lipschitz curve indep of t.

Aim: Study $V : [0,T]\times X \longrightarrow (-\infty,\infty]$.

$$\text{Hess } V \ge \tfrac{1}{2}\partial_t g_t$$

metric tensor, we need to define it.

Apply it to: • $X = \mathcal{P}_2$, $V_t(\mu) = \text{Ent}(\mu | m_t)$ for given (X, d_t, m_t).
• $t \mapsto -t$.

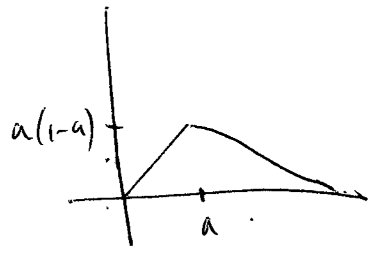
Infinitesimal action for Lipschitz curve $\gamma : [0,1] \longrightarrow X$ at a.e. $a \in [0,T]$

$$g_t^a(\gamma) = \lim_{b\to a} \left|\frac{d_t(\gamma^a,\gamma^b)}{b-a}\right|^2 = \text{square of metric derivative of } \gamma \text{ w.r.t. } d_t.$$

Action of γ: $\mathcal{A}_t(\gamma) = \lim_k 2^k \sum_{i=1}^{2^k} d_t^2(\gamma^{i2^{-k}}, \gamma^{(i-1)2^{-k}}) = \int_0^1 g_t^a(\gamma)\, da$.

$$\left(\text{Similar } \text{length}(\gamma) = \int_0^1 g_t^a(\gamma)^{\frac{1}{2}}\, da\right)$$

$X^{a,b}$ = Green fnctn on $[0,1]$.



$$b \mapsto X^{a,b}, \quad \int_0^1 X^{a,b}\, db = \tfrac{1}{2} b(1-b).$$

$$\mathcal{A}_t^a(\gamma) = \int_0^1 X^{a,b} g_t^b(\gamma)\, db.$$

Suppose stronger Log-Lip bound:

$$\left|\log \frac{d_t(x,y)}{d_s(x,y)}\right| \le C^*|t-s| , \quad \forall t,s, x,y$$

Consequently, $t \mapsto d_t^2(x,y)$ is diff. a.e. in t.

$$\forall s>r, \quad d_s^2(x,y) - d_r^2(x,y) = \int_r^s \partial_t^+ d_t^2(x,y)\,dt .$$

where $\partial_t^+ u(t) := \limsup_{s\to t} \frac{1}{s-t}[u(s)-u(t)]$.

$\Rightarrow$ $\left|\log \frac{g_t^a}{g_s^a}\right| \le 2C^*|t-s|$ $\forall$ Lipschitz γ : $t \mapsto g_t^a(\gamma)$.

Fix a d_t-geodesic γ, $\forall s>t$,

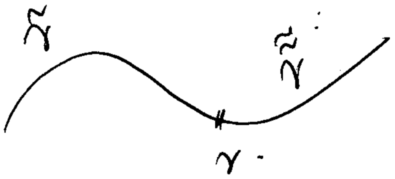
$$2^k \sum_{i=1}^{2^k} \frac{1}{s-t}\left[d_s^2\left(\gamma^{i2^{-k}}, \gamma^{(i-1)2^{-k}}\right), d_t^2\left(\gamma^{i2^{-k}}, \gamma^{(i-1)2^{-k}}\right)\right]$$

increasing in k.

$\Rightarrow$ $\underbrace{2^k \sum \partial_t^+ d_t^2\left(\gamma^{i2^{-k}}, \gamma^{(i-1)2^{-k}}\right)}$ is also increasing in k.

Also, $\partial_t^+ d \le C^* d$. so $\nearrow$ $\le 2C^* d_t^2(\gamma^0,\gamma^1)$.

$\Rightarrow$ $h_t(t) = \lim_{k\to\infty}$ $\cancel{\quad}$



$\Rightarrow$ Increases Lip. as $b \mapsto h_t(\gamma|_{[0,b]})$.

$\Rightarrow$ $\exists h_t^a(\gamma)$: $h_t(\gamma) = \int_0^1 h_t^a(\gamma)\,da$.

is $\le 2C^* g_t^a$ for a.e.

Assume $V:[0,T]\times X \to (-\infty,\infty]$, uniformly lower bound, measurable in (t,x), lsc in x $(\forall t)$, bound for $|\partial_t \log d_t| \le C^*$.

Def. V is dynamically convex $\iff$ $\forall t\in[0,T]$, $\forall x^0,x^1\in X$ with $V_t(x^0)$, $V_t(x^1) < \infty$, $\exists$ d_t-geodesic connecting x^0,x^1, $\forall a\in[0,\tfrac12]$:

$$V_t(\gamma^0)+V_t(\gamma^1)-V_t(\gamma^a)-V_t(\gamma^{1-a}) \ge \frac{a}{2}\,\partial_t^+ d_t^2(x^0,x^1) - C^* a^2 d^2(x^0,x^1).$$

If the above holds for a.e. $t\in[0,T]$, then V called a.s. dyn. convex.

V is called backward dynamically convex if the same with $\partial_t^+ d_t^2$ replaced by $\partial_t^- d_t^2$, $\partial_t^- u_t = \liminf_{s\nearrow t} \frac{1}{t-s}[u(t)-u(s)]$

Some consequences.

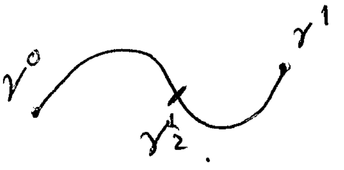
(I) $\partial_a V(\gamma^{1-}) - \partial_a V(\gamma^{0+}) \ge \frac12 \partial_t^+ d_t^2(x^0,x^1)$.

where $\partial_a V(\gamma^{1-}) = \limsup_{a\searrow 0} \frac{1}{a}\left(V(\gamma^1)-V(\gamma^{1-a})\right)$.

$\partial_a V(\gamma^{0+}) = \liminf_{a\searrow 0} \frac{1}{a}\left(V(\gamma^a)-V(\gamma^0)\right)$.

(II) Apply $(*)$ to $a=\frac12$:

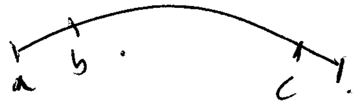
$$\tfrac12 V(\gamma^0)+\tfrac12 V(\gamma^1)-V(\gamma^{\frac12}) \ge -C_1 d^2(\gamma^0,\gamma^1).$$



$\Rightarrow$ $\exists$ geodesic $\hat\gamma$ connecting x^0,x^1 s.t. $a\mapsto V_t(\hat\gamma_t^a)$ is $(-C_1)$-convex.

Assume $\forall t$: $x \mapsto V_t(x)$ is $(-C)$-convex along every d_t geodesic.
(Ex: follows from (II) + inf Hilbertian).

Then, for each d_t geodesic,
(III). $\forall\, b \in [0,1]$, ~~$\partial_a V(\gamma^{b})$~~



$$\partial_a(\gamma^{b-}) - \partial_a V(\gamma^{b+}) \geq \frac{1}{2}\int_b^c h_t^a(\gamma)\,da \geq \frac{1}{2}\partial_t^+ d_t^2(\gamma^b,\gamma^c).$$

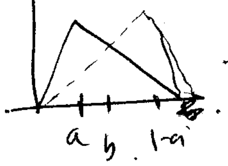
Pf. Divide $[0,1]$ into 2^k intervals, apply (I) in each $[(i-1)2^{-k}, i2^{-k}]$ interval, and prove it there, left derivative smaller than right by convexity.

(IV) In the sense of measures in $[0,1]$
$$\partial_a^2 V(\gamma^a) \geq \frac{1}{2}h_t^a(\gamma) \quad \text{for a.e. } a.$$

(V). $\forall a \in [0,1]$, $(1-a)V(\gamma^0) + aV(\gamma^1) - V(\gamma^a) \geq \frac{1}{2}\int \chi^{a,b} h^b(\gamma)\,db$.

$\Rightarrow V(\gamma^0) + V(\gamma^1) - V(\gamma^a) - V(\gamma^{1-a}) \geq \frac{a}{2}\int_0^1 \Xi^{a,b}\, h^b(\gamma)\,db$.

by convex.



for $\chi^{a,b}$.

$= \int_0^1 h^b(\gamma)\,db - \int_0^1 (1-\Xi^{a,b})h^b(\gamma)\,db$

$\geq \partial_t^+ d_t^2(x^0,x^1) - 2C^*\int (1-\Xi^{a,b})\, g^b(\gamma)\,db$.

$= \text{---}\,{/\!/}\,\text{---} - 2C^* d_t^2(x^0,x^1)\,a$.

Prop V is a.e. dyn. convex $\iff$ $\forall x^0, x^1.\ \forall\, 0 \leq r < s \leq T$:
$\exists$ geodesics $(\cdot\, d_t)$. γ_t.

$$\int_r^s \left[V_t(\gamma_t^0) + V_t(\gamma_t^1) - V_t(\gamma_t^a) - V_t(\gamma_t^{1-a})\right]$$

$$\geq \frac{1}{2}\left[d_s^2(x^0, x^1) - d_r^2(x^0, x^1)\right]$$

$$- C^* a^2 \int_0^S d_t(x^0, x^1)\, dt.$$

Def. V is called dyn. N-convex $\iff$
$\forall t, x^0, x^1$ s.t. $V_t(x^0), V_t(x^1) < \infty$; $\exists d_t$ geodesic
s.t. $\forall a \in [0, a^*]$ with $a^* = a^*(V_t(x^0), V_t(x^1))$, $C^*(N)$.

$$\frac{1}{a}\left[\Phi_N\left(V_t(x^0) - V_t(\gamma_t^a)\right) + \Phi_N\left(V_t(x^1) - V_t(\gamma_t^{1-a})\right)\right]$$

$$\geq \frac{1}{2}\partial_t^+ d_t^2(x^0, x^1) + \frac{1}{N}\left(V_t(x^0) - V_t(x^1)\right)^2$$

where $\Phi_N(x) = x + \frac{1}{N}x^2$, $\qquad - C^* a\, d_t^2(x^0, x^1)$.

and $a^* = \frac{1}{2}\min\left\{1, N\left[V(x^0) - V(x^1) + C^* d^2(x^0, x^1)\right]^{-1}\right\}$.

Consequences:

(I) For $a \to 0$, $-\partial_a V(\gamma^{0+}) + \partial_a V(\gamma^{1-}) \geq \frac{1}{2}\partial_t^2 d_t^2(x^0, x^1)$

$$+ \frac{1}{N}\left(V_t(x^0) - V_t(x^1)\right)^2.$$

(II) If V is $(-C)$-convex along all geodesics, partition small pieces and sum to get.

$$-\partial_a V(\gamma^{b-}) + \partial_a V(\gamma^{c-}) \geq \frac{1}{2}\int_b^c h_t^a(\gamma)\, da$$

$$+ \frac{1}{N}\int_b^c |\partial_a V(\gamma)|^2\, da.$$

(III). $\partial_u^2 V(\gamma) \geq \frac{1}{2} h_t^a(\gamma) + \frac{1}{N} |\partial_a V(\gamma)|^2$.

(IV) $\forall a \in [0,1]$.

$$\frac{1}{a}\left[\begin{matrix}\cancel{(1-a)}V(\gamma^0) + a V(\gamma^1) - V(\gamma^a) \\ V(\gamma^{1-a}) + \end{matrix}\right] \geq \frac{1}{2}\int \boxtimes^{a,b} h_t^a(\gamma) \quad \gtrsim \text{\_\_\_}$$

$$\geq \frac{1}{N}\int_0^1 \Xi^{ab}\, |\partial_t V(\gamma^b)|^2\, db.$$

C.S.

$$\geq \frac{1}{N}\int_0^{1-a} |\partial_b V|^2\, db.$$

$$\geq \frac{1}{N(1-2a)}\left(\int_a^{1-a} \partial_b V(\gamma^b)\, db\right)^2$$

$$\geq \frac{1}{N(1-2a)}\left(V(\gamma^{1-a}) - V(\gamma^a)\right)^2$$] not stable under convergence, not applicable to VEEnt.

$$\geq \frac{1}{N}\left[\left(V(\gamma^1) - V(\gamma^0)\right)^2 - \frac{1}{a}\left(V(\gamma^1) - V(\gamma^{1-a})\right)^2 - \frac{1}{a}\left(V(\gamma^0) - V(\gamma^a)\right)^2\right].$$

(V). $n \mapsto \Phi(n) = n + \frac{n^2}{N}$ is increasing for $n \geq -N/2$,

$\forall a \leq a^*$, $V(\gamma^0) - V(\gamma^a) \geq -N/2$, $V(\gamma^1) - V(\gamma^{1-a}) \geq -N/2$.