Super Ricci Flow for MMS. — Sturm. 30/03/2015.

Classical: Ricci flow for (M, g_t), $Ric(g_t) = -\frac{1}{2}\partial_t g_t$

Super Ricci flow. $Ric(g_t) \geq -\frac{1}{2}\partial_t g_t$.

Ex Static: $\partial_t g_t = 0 \Rightarrow Ric(g_t) \geq 0$.

* Super Ricci flow preserves GH-limits!

In abstract setting, interested in N-Ricci flow:

$Ric_g^N \geq -\frac{1}{2}\partial_t g_t$. $\leftarrow$ ~~some~~ finsid, completeness here.

General: $(X, d_t, m_t)_{t\in[0,T]}$, or $(X, \Gamma_t, m_t)_{t\in[0,T]}$.

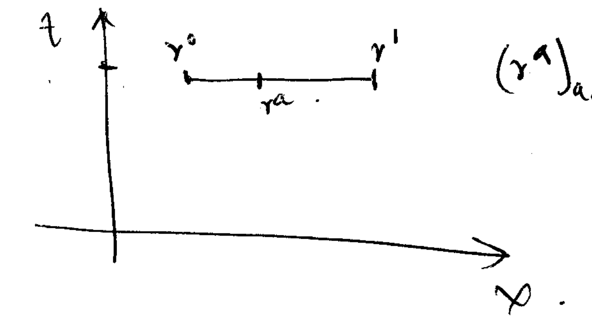
(I) $\left|\log \frac{d_t}{d_s}\right| \leq C.\ \forall t,s$. or even $\leq C^*|t-s|$.

(II). $\left|\log \frac{dm_t}{dm_s}\right| \leq C$. or Lip cts in t, x.

- Consequence of (I): $Ric(X, d_t, m_t) \geq -2C^*\ \ \forall t$.
  ie $CD(-C^*, \infty)$.

- If $\forall t$. (X, d_t, m_t) inf Hilbertian,
  OT-calculus $\Leftrightarrow$ Bakry-Émery Calculus. $(\forall t)$

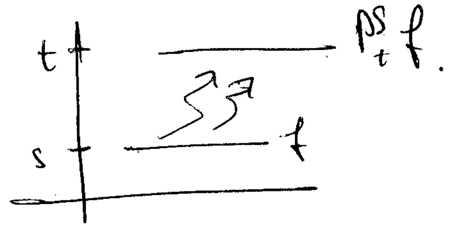
Notation



$(\gamma^a)_{a\in[0,1]} \leftarrow$ t fixed ; d_t geodesic.

Under these assumptions, get: Heat flow on functions.
gradient flow of energy

$$\partial_t u_t = \Delta_t u_t.$$
$$u_s = f, \qquad s<t.$$



- Heat flow on measures; defined via duality.

$$\int f \, d\underbrace{\hat{P}^s_t \mu}_{\hat{P}^s_t \text{ action on measures}} := \int P^s_t f \, d\mu.$$

This is a backward gradient flow of Ent.

- Density point of view:

$$\partial_s \mu_s = -\hat{L}_s \mu_s.$$

If $\mu_s = u_s m_s$, $\partial_s u_s = -L_s u_s - \left(\frac{\partial f_s}{\partial s}\right) u_s$,

$$e^{-f_s} m_s = e^{-f_t} m_t.$$

Warm up: hyper Ricci flow for Γ-calculus.

Setting: Family of diffusion operators L_t, $t \in [0,1]$ defined on algebra $\mathcal{A}$ of functions.

$$\Gamma_t(u) = \tfrac{1}{2} L_t(u^2) - u L_t u.$$

$$\Gamma_{2,t}(u) = \tfrac{1}{2} L_t(\Gamma_t(u)) - \Gamma_t(u, L_t u).$$

Assume L_t are local, have cts. paths.

Ricci operator: $R_t(u)(x) = \inf \{ \Gamma_{2,t}(u+v)(x) : v \in \mathcal{A}_x^0 \}$.

$\mathcal{A}_x^0 = \{ v = \psi(u_1, \dots, u_k) : \psi$ smooth, $u_i \in \mathcal{A}$,

$\partial_i \psi(u_1, \dots, u_k)(x) = 0 \quad \forall i \}$.

From Bochner: $\Gamma(u) = |D^2 u|^2_{HS} + Ric(\nabla u)$.

Want $\tilde{u}$ in place of u so that $D^2 \tilde{u} = 0$.

This is the def^n for $\mathcal{A}_x^0$.

Def^n. $(L_t)_t$ is a hyper Ricci flow

$\iff \Gamma_{2,t}(u) \geq \tfrac{1}{2} \partial_t \Gamma_t(u). \quad \forall u \in \mathcal{A}$.

Prop. $\iff R_t(u) \geq \tfrac{1}{2} \partial_t \Gamma_t(u)$.

Remark. Ricci flow $R_t(u) = \tfrac{1}{2} \partial_t \Gamma_t(u). \quad \forall u$.

$\not\Leftrightarrow \Gamma_{2,t}(u) = \tfrac{1}{2} \partial_t \Gamma_t(u). \quad \forall u$

Def$^{\underline{n}}$. N-super Ricci flow $\Longleftrightarrow$

$$\Gamma_{2,t}(u) - \frac{1}{N}(L_t u)^2 \geq \frac{1}{2}\partial_t \Gamma_t(u). \quad \forall u \in \mathcal{A}.$$

Remark: (I) Until now, no measures.
(II) Works on non-symmetric operators as long as you have a way to produce semi groups.

Assume $\exists\, P_t^s$ operators on $\mathcal{A}$ $(\forall s<t)$. Algebra.

$$P_t^t u = u, \quad P_t^r(P_r^s u) = P_t^s u, \quad (P_t^s u)^2 \leq P_t^s(u^2).$$

$s \mapsto P_t^s u$, $t \mapsto P_t^s u$ continuous.

$$\partial_s P_t^s u = -P_t^s(L_s u). \quad \Big| \text{ interacts with Heat well.}$$
$$\partial_t P_t^s u = L_t P_t^s u.$$

Th$^{\underline{m}}$. Under same assumptions, including TFAE:

(I). $\Gamma_{2,t}(u) \geq \frac{1}{2}\partial_t \Gamma_t(u)$.

(II). $\Gamma_t(P_t^s u)(x) \leq P_t^s \Gamma_s(u)(x)$

Pf. Trick often used in Γ-calculus arguments:

$r \in [s,t]$, $q_r := P_t^r \Gamma_r(P_r^s u)$, differentiate.

$$\partial_r q_r = P_t^r\big(-L_r \Gamma_r(P_r^s u) + \partial_r \Gamma_r(P_r^s u) + 2\Gamma_r(L_r P_r^s u, P_r^s u)\big).$$

$$\overset{\Gamma_2 \text{ def}}{=} P_t^r\big(\underbrace{-2\Gamma_{2,r}(P_r^s u) + \partial_r \Gamma_r(P_r^s u)}_{\leq 0}\big).$$

≤ 0 of super Ricci flow. and ≤ 0

$\Rightarrow q_s \geq q_t$. Thus (I) $\Rightarrow$ (II). Converse similar.

Thm TFAE:

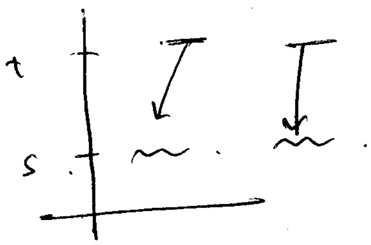
(I) $\Gamma_{2,t}(u) - \frac{1}{N}(L_t u)^2 \geq \frac{1}{2}\partial_t \Gamma_t(u)$.

(II) $\Gamma_t(P_t^s u) + \frac{2}{N}\int_s^t (P_t^r L_r P_r^s u)\,dr \leq P_t^s P_s u$.

Pf Same q_r as before,

$$\partial_r q_r = P_t^r\left(-2\Gamma_{2,r}(u) + \partial_r \Gamma_r(u)\right)$$
$$\leq -\frac{2}{N} P_t^r\left((L_r v)^2\right)$$
$$\leq -\frac{2}{N}\left(P_t^r L_r v\right)^2.$$

$$\Rightarrow q_s \geq q_t + \int_s^t -\frac{2}{N}(P_t^r L_r v)^2\,dr. \quad \text{Setting } v = P_r^s u.$$



Prop (II) $\xrightarrow{(N=\infty)}$ $W_s(\hat{P}_t^s \mu, \hat{P}_t^s \nu) \leq W_t(\mu, \nu)$.

(2-Wasserstein w.r.t. d_s, d_t etc).

Existence of heat propagators.

Setting: Strongly local Dirichlet forms $\mathcal{E}_t$ on $L^2(X, m_t)$.

- $\left|\mathcal{E}_s(u)/\mathcal{E}_t(u)\right| \leq c$ $(\forall s,t \in I \subseteq [0,T])$,

$$\left|\frac{dm_t}{dm_s}\right| \leq C.$$

Assume $\exists$ Γ-operators $\left|\Gamma_s(u)(x)/\Gamma_t(u)(x)\right| \leq C$.

and $m_t = e^{-f_t} m_0$.

• Important assumption : $\exists C$, $f_t \in \mathcal{D}(\mathcal{E}) := \mathcal{D}(\mathcal{E}_t) = \mathcal{D}(\mathcal{E}_s)$

This is automatic for assumption 1, because $|\mathcal{E}_t(u)/\mathcal{E}_s(u)| \leq C$.

$$\Gamma_0(f_t)(x) \leq C \quad \forall t, x.$$

Thm. $\exists$ concept of weak sol^n of $\partial_t u = L_t u$ in $(s,T) \times X$.

• $\forall h \in L^2(X, m_0)$, $\exists!$ sol^n of with $u_s = f$.

$$u \in \mathcal{F}_{[s,T]} = L^2((s,T) \to \mathcal{F}) \cap H^1((s,T) \to \mathcal{F}^*)$$
$$\subset C^0([s,T] \to \mathcal{H})$$
$$\mathcal{H} = L^2(X, m_0).$$

Need Thm.

is what allows us to obtain $u_s = f$ i.e, at endpoint. Need continuity guaranteed by this theorem.

Pf.
$$\int \partial_t u \cdot v \, dm_0 = \int L_t u \cdot v \, dm_0 \quad (L_t \text{ not symm. w.r.t. to } m_0)$$
$$= \int L_t u \cdot v \, e^{f_t} \cdot e^{-f_t} dm_0.$$
$$= \int L_t u \cdot (e^{f_t} v) \, dm_t.$$
$$= \int \Gamma_t(u, v e^{f_t}) \, dm_t.$$
$$= \int [\Gamma_t(u,v) + v \cdot \Gamma_t(u, f_t)] \cdot dm_0.$$
$$= \mathcal{E}_t^0(u,v), \quad \text{bilinear, non-symm.}$$

Lions-Magenes $\Gamma_t(f_t) \le C \Rightarrow \exists!\ P_t^s u$ & continuity in t.

Gives: $\partial_t P_t^s u = L_t P_t^s u$.

Need $\partial_s P_t^s u = -P_t^s L_s u$, need continuity in s.

for this, assume $|f_t(x) - f_s(x)| \le c|t-s|$.

independent on x.

Consider: $L_t u - (\partial_t f_t) u = -\partial_t u$.

$u \ge 0 \Rightarrow P_t^s u \ge 0$.

$u \le 1 \Rightarrow\ \le 1$.

Assume: • $|\Gamma_t / \Gamma_s| \le C$, $|m_t / m_s| \le C$.

• f is Lip in t and x, $t \mapsto \log c_t$ is Lip. in t (unif. in x)

• $\exists\, C_P, C_D$, $\forall t$: Poincaré (C_P), Doubling (C_D).

$\Rightarrow$ Lierl + Saloff-Coste, $(t,x) \mapsto P_t^s u(x)$ satisfies parabolic Harnack.

$(s,x) \mapsto P_t^s u(x)$ " "

$\exists$ density $p_t^s(x,y)$ s.t. $P_t^s u(x) = \int u(y)\, p_t^s(x,y)\, dm_s(y)$

$p_t^s(x,y)$ Hölder in each argument.

Assumptions true for uniform lower Ricci bound $- C^\infty (\forall t)$,

+ inf Hilbert.