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Let $t \mapsto A_t \in \mathbb{R}^{d\times d}_{>0}$.

Then, $\partial_t A_t^2 = A_t A_t' + A_t' A_t$

$\partial_t A_t^3 = A_t^2 A_t' + A_t A_t' A_t + A_t' A_t^2$.

Exercise: $\partial_t e^{A_t} = ?$

Entropy gradient flows on non-comm. probability.

(I) Two views on heat flow

- L^2-theory: - Gauss space $L^2(\mathbb{R}^n, \gamma)$ ← std Gauss.
  - Dirichlet energy $\mathcal{E}_\gamma(f) = \int |\nabla f|^2 \, d\gamma$.

  ⇒ Grad flow for $\mathcal{E}_\gamma$ in $L^2(\mathbb{R}^n, \gamma)$ is the Ornstein-Uhlenbeck eq. $\partial_t f = -\mathcal{L} f$.

$$-\mathcal{L} f(x) = \Delta f(x) - \langle x, \nabla f(x) \rangle.$$

- OT-theory: - 2-Wass. space $(\mathcal{P}(\mathbb{R}^n), W_2)$.
  - Rel entropy. $\mathrm{Ent}_\gamma(\mu) = +\int \rho \log \rho \, d\gamma$ $\rho = \frac{d\mu}{d\gamma}$.

  → Grad flow of Ent_γ w.r.t. to W_2 is the same. O-U eq.

(II) Quantum-Mech pt of view: (on $L^2(\mathbb{R}^n)$).

- Two particles - bosons and fermions, ~~treat~~ with diff symm. properties. We look at

$$\widetilde{\mathcal{F}}^n_{sym} = \bigoplus_{m\geq 0} \underbrace{(\mathbb{R}^n)^{\otimes m}_{sym}}_{\text{to account for annihilation/creation}}.$$ Symmetric Fock space.

↗ Bosonic, symm.

- Wiener: canonical isometry: $I : L^2(\mathbb{R}^n, \gamma) \longrightarrow \tilde{F}^n_{sym}$.
  - Hermite polynomials are dense in $L^2(\mathbb{R}^n, \gamma)$, let $\{H_m\}$ be the set of Hermite poly.

$$H_{m_1}(x_1) \cdots H_{m_n}(x_n) \longmapsto \underbrace{P_{sym}}(e_1^{\otimes m_1} \otimes \cdots \otimes e_n^{\otimes m_n})$$

projection to sym terms.

$\{e_i\}_{i=1}^n$ std basis for $\mathbb{R}^n$.

$$e_i^{\otimes m_i} = \underbrace{e_i \otimes e_i \otimes \cdots \otimes e_i}_{m_i \text{ times}}$$

- $N = I \circ \mathcal{L} \circ I^{-1}$ is the bosonic number operator.

$$N\xi = m\xi \quad \forall \xi \in (\mathbb{R}^n)^{\otimes m}_{sym}.$$

(III) <u>Fermionic counterpart.</u>

- $\tilde{F}^n_{asym} = \bigoplus_{m \geq 0} (\mathbb{R}^n)^{\wedge m}$ (anti-sym. tensors).

$$= \bigoplus_{m=0}^{n} (\mathbb{R}^n)^{\wedge m} \quad \text{by anti-sym.}$$

- Fermionic number operator: $N\xi = m\xi$ for $\xi \in (\mathbb{R}^n)^{\wedge m}$.

- there exists a counterpart, but this time, need to work with operators (matrices) rather than functions.

<u>Def</u>. Let $Q_1, \dots, Q_n$ be self-adjoint operators on a Hilbert space satisfying:

$$i \neq j \Rightarrow Q_i Q_j = -Q_j Q_i \quad \text{otherwise } Q_i^2 = I.$$

The Clifford algebra is given by $\mathcal{C}^n := \operatorname{span}\{Q_1^{\alpha_1} \cdots Q_n^{\alpha_n} \; \alpha_i \in \{0,1\}\}$

- Fermionic Wiener isometry: $I: \cancel{\mathcal{C}}^n \longrightarrow \mathcal{F}^n_{asym}$.

$$Q_{\underline{\alpha}} \longmapsto P_{asym}(e_1^{\otimes \alpha_1} \otimes \cdots \otimes e_n^{\otimes \alpha_n}) \qquad P_{asym}\ \text{~~anti-sym~~ projection onto asym.}$$

<u>Rem.</u> Anti-sym. nature of fermions means you cannot expect scalars.

(IV) Analysis on $\mathcal{C}^n$.

(1) N.C. integration, def $\tau: \mathcal{C}^n \to \mathbb{C}$ by

$$\tau(Q_{\underline{\alpha}}) := \begin{cases} 1 & \underline{\alpha} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\rightsquigarrow$ τ is a trace; positive linear functional s.t.

$$\tau(AB) = \tau(BA).$$

τ induces a scalar product on $\mathcal{C}$: $\langle A, B\rangle = \tau(\cancel{AB})$.

(2). N.C. analogue of differentiation; define $\partial_i: \mathcal{C} \to \mathcal{C}$ by

$$\partial_i A = \tfrac{1}{2}(Q_i A - \Pi(A) Q_i). \qquad \text{(almost commutator).}$$

where $\Pi(Q_{\underline{\alpha}}) = (-1)^{|\underline{\alpha}|} Q_{\underline{\alpha}}$ (because ~~its~~ really an anti-commutator).

Define $\mathcal{E}: \mathcal{C} \times \mathcal{C} \to \mathbb{C}$, $\mathcal{E}(A,B) = \sum_i \langle \partial_i A, \partial_i B\rangle$.

$\Rightarrow$ the operator associated to $\mathcal{E}$ is the number operator.

$$\mathcal{E}(A,B) = \langle \mathcal{L} A, B\rangle \quad \forall A, B, \quad \mathcal{L} = I^{-1} \circ N \circ I.$$

Gross '75, Carlen-Lieb '93: Fermionic log-Sobolev inequality.

$$\tau(A^* A \log(A^* A)) \leq 2 \cdot \mathcal{E}(A^*, A), \quad \forall A \in \mathcal{C}$$

~~(Fermionic)~~ Classic log-Sob. $\int f^2 \log f^2 \, d\gamma \leq 2 \int |\nabla f|^2 d\gamma$, if $f \geq 0$, $\int f^2 d\gamma = 1$.

Q. Is there a version of Wasserstein? Classic log-Sob. can be proved via mass-transport.

<u>Def.</u> Let $\mathcal{P} = \{\rho \in \mathcal{C}^n : \tau(\rho) = 1,\ \rho \geq 0\}$.

For $\rho_0, \rho_1 \in \mathcal{P}$; define

$$W(\rho_0, \rho_1)^2 = \inf \left\{ \int_0^1 \sum_j \langle \rho_t \circ \partial_j \psi, \partial_j \psi \rangle dt \right\}$$

s.t. $\partial_t \rho + \sum_j \partial_j^* (\rho \circ \partial_j \psi) = 0$, $\rho|_{t=0,1} = \rho_{0,1}$.

(Compare with B-B formula: for $\rho_0, \rho_1 \in \mathcal{P}(\mathbb{R}^n)$;

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^n} |\nabla \psi_t(x)|^2 d\rho_t(x)\, dt \text{ s.t. } \partial_t \rho + \nabla \cdot (\rho \nabla \psi) = 0,\ \rho_{t=0,1} = \rho_{0,1} \right\}.)$$

Multiplication is non-commutative here, so

$$\boxed{\rho \circ A := \int_0^1 \Pi(\rho)^{1-\alpha} A \rho^\alpha \, d\alpha .} \quad (*)$$

<u>Thm</u>: [Carlen-M.] The fermionic O.U. eq. $\partial_t \rho = -\mathcal{L}\rho$ is the gradient flow of the Neumann entropy

$$\mathrm{Ent}(\rho) = \tau(\rho \log \rho)$$

w.r.t. to W.

why $(*)$?

<u>Prop</u>: $\forall \rho \in \mathcal{P}_{>0}$: $\partial_j \rho = \rho \circ \partial_j \log \rho$.

Back to exercise: let $t \mapsto A_t \in \mathbb{R}^{d\times d}_{sym}$.

$$\partial_t A^n = \sum_{k=0}^{n-1} A_t^k \cdot \partial_t A_t \, A_t^{n-k-1}. \quad \forall n$$

Put $A_t = (\rho_t)^{\frac{1}{n}}$ for a curve $\rho_t > 0$.

$$\partial_t \rho_t = \sum_{k=0}^{n-1} \rho_t^{k/n} \partial_t (\rho_t^{\frac{1}{n}} - I) \cdot \rho_t^{1-\frac{k+1}{n}}$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} \rho_t^{k/n} \underbrace{\partial_t (\rho_t^{\frac{1}{n}} - I)}_{\frac{1}{n}}\, \rho_t^{1-\frac{k+1}{n}} \xrightarrow{n\to\infty} \int_0^1 \rho_t^{\alpha} \partial_t \log \rho_t \, \rho_t^{1-\alpha} d\alpha$$

$$\Rightarrow \partial_t e^{mA_t} = \int_0^1 e^{\alpha A_t} \cdot A_t' e^{(1-\alpha)A_t}\, d\alpha.$$

$\boxed{\times}$ B

- In the process of passing to limit, we fix a curve $\rho_t > 0$ and then define $A_t = (\rho_t)^{\frac{1}{n}}$, so that we can apply formula for $\partial_t A_t^n$. Then, we obtain an expression where the RHS has n, and $\partial_t \rho$ is indep of n. That's why we can pass to limit.

~~Rem~~

My note: the underlying Hilbert space H for which the Q. Lens fn. does not matter, this is simply a representation, because we always set

$$\mathcal{L} = I^{-1} \circ \mathcal{M} \circ I$$

where $\mathcal{M}$ is fixed on $\mathcal{P}^n$ Lebesgue. and the I changes when H changes. So I can be thought of as a representation.