Starting point (Ricci curv.).

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Th^m. [Cordero, McCann, Schmuckenschläger, Otto-Villani, von Renesse, Sturm].

Let (M,g) be Riem mfld. Then TFAE:

(I) $Ric \geq \kappa g$ on M.

(II) $\mathcal{H}(\rho) = \int \rho \log \rho$ is κ-convex along W_2-geodesics.

Ie, $\forall$ geodesics ρ_t.

$$\mathcal{H}(\rho_t) \leq (1-t)\mathcal{H}(\rho_0) + t\mathcal{H}(\rho_1) - \kappa/2 \, t(1-t) W_2^2(\rho_0, \rho_1).$$

$\Rightarrow$ Metric-measure spaces with lower Ricci bounds.

But this fails for discrete:

Ex. $\mathcal{X} = \{0,1\}$, $(P\{0,1\}, W_2) \cong ([0,1], d(x,y) = \sqrt{|x-y|})$.

Exercise: All geodesics in ↑ are const.

Recall: discrete setup:

- $\mathcal{X}$ finite set.
- $\mathcal{L}\psi(x) = \sum_y Q(x,y). \psi(y) - \psi(x)$ generator of Markov chain. $Q(x,y) \geq 0$. ~~Rates~~.
- Assume π: reversible measure, ie balanced eq^n.

$$\pi(x) Q(x,y) = \cancel{Q(x} \; \pi(y) Q(y,x) =: w(x,y).$$

Def. For $\rho_0, \rho_1 \in \mathcal{P}(\mathcal{X})$, $\quad \mathcal{W}(\rho_0,\rho_1)^2 = \inf \Big\{ \frac{1}{2} \int_0^1 \sum_{x,y} (\psi_t(x) - \psi_t(y))^2 \, \hat{\rho}_t(x,y)\, \omega(x,y)$

s.t. $\partial_t \rho(x) + \sum_y (\psi_t(x) - \psi_t(y)) \hat{\rho}_t(x,y) Q(x,y) = 0 \Big\}$

$\wedge$: define function on edges if given a function on vertices.

$$\rho_\cdot|_{t=0,1} = \rho_0, \rho_1, \qquad \hat{\rho}(x,y) := \int_0^1 \rho(x)^{1-s} \cdot \rho(y)^s \, ds .$$

Def. Let $k \in \mathbb{R}$. We say that $\mathrm{Ric}(\mathcal{X}, Q, \pi) \geq k$ if $\mathcal{H}$ is k-convex along $\mathcal{W}$-geodesics.

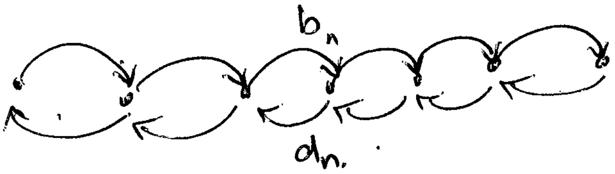
Prop (discrete Bakry-Emery). If $\mathrm{Ric}(\mathcal{X}, Q, \pi) \geq k > 0$, then the modified log-Sobolev holds:

(MLSI) $$\mathcal{H}(\rho) \leq \frac{1}{2k} \mathcal{I}(\rho), \qquad (\mathcal{I}(\rho) - \text{Fischer info}).$$

$$\mathcal{I}(\rho) := \mathcal{E}(\rho, \log \rho) = \frac{1}{2} \sum_{x,y} (\rho(x) - \rho(y))(\log \rho(x) - \log \rho(y)) Q(x,y) \pi(x)$$

Rmk. (MLSI) $\iff \mathcal{H}(e^{t\mathcal{L}}\rho) \leq e^{-2kt} \mathcal{H}(\rho)$.

Examples: • Mielke: 1-D birth-death chains.



Agrees with cts case: i.e., discretisation of Fokker-Planck eqn.

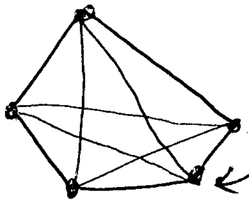
• Erbar – M: tensorisation result:

If $\mathrm{Ric}(\mathcal{X}_i, Q_i, \pi_i) \geq \kappa_i \quad i=1,2,$

$$\Rightarrow \mathrm{Ric}(\mathcal{X}_1 \times \mathcal{X}_2, Q_{prod}; \pi_1 \otimes \pi_2) \geq \min\{\kappa_1, \kappa_2\}.$$

• Erbar – M – Tetali, Fathi – M:

Bernoulli-Laplace model.



• Complete graph, n sites, k particles.

• randomly select site and particle.

• at rate 1, a random particle jumps to a random empty site.

Thm. $\mathrm{Ric} \geq \dfrac{n+2}{2k(n-k)}$.

How to show lower Ricci bounds?

Ctn case: on a Riem mfld (Purely formal, not proof).

geodesic eq^{n}s for W_2:
$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \nabla \psi) = 0 \\ \partial_t \psi + \frac{1}{2} |\nabla \psi|^2 = 0. \end{cases}$$

Then;
$$\partial_t H(\rho) = \int (1+\log \rho)\, \partial_t \rho = -\int (1+\log\rho)\, \nabla \cdot (\rho \nabla \psi).$$
$$= \int \nabla \log \rho \cdot \rho \nabla \psi = \int \nabla \rho \cdot \nabla \psi$$

$$\partial_t^2 H(\rho) = -\int \partial_t \rho \cdot \Delta \psi - \int \Delta \rho \cdot \partial_t \psi$$
$$= \int \nabla \cdot (\rho \nabla \psi) \Delta \psi + \frac{1}{2} \int \Delta \rho \cdot |\nabla \psi|^2 .$$
$$= \int \left(-\langle \nabla \psi, \nabla \Delta \psi \rangle + \frac{1}{2} \Delta |\nabla \psi|^2 \right) d\rho$$
(Bochner Weitzenböck)
$$= \int |\nabla^2 \psi|^2 + \mathrm{Ric}(\nabla \psi, \nabla \psi)\, d\rho$$

Discrete case:

Prop: The geodesic equations in $(\mathcal{P}(\mathcal{X}), \mathcal{W})$ are:

$$\begin{cases} \partial_t \rho(x) + \sum_y (\psi_t(x) - \psi_t(y)) \hat{\rho}_t(x,y) Q(x,y) = 0. \\ \partial_t \psi(x) + \frac{1}{2} \sum_y \partial_1 \theta(\rho(x), \rho(y)) (\psi_t(x) - \psi_t(y))^2 Q(x,y) = 0 \end{cases}$$

~~Recall~~ Recall $\theta(s,t) = \int_0^1 s^{1-\alpha} t^{\alpha} \, d\alpha$.

Slightly harder than before, b/c this is a coupled system of eq^{n}s.

Then $\partial_t H(\rho) = \langle \nabla \rho, \nabla \psi \rangle_{L^2(\mathcal{E},\pi)}$. ($\nabla$ = discrete grad).

$$\partial_t^2 H(\rho) = -\langle \hat{\rho} \nabla \psi, \nabla \mathcal{L} \psi \rangle_{L^2(\mathcal{E},\pi)} + \frac{1}{2} \langle \hat{\mathcal{L}}_\rho, (\nabla \psi)^2 \rangle_{L^2(\mathcal{E},\omega)}.$$

$$\hat{\mathcal{L}}_\rho(x,y) = \partial_1 \theta(\rho(x), \rho(y)) \mathcal{L} \rho(x) + \partial_2 \theta(\rho(x), \rho(y)) \mathcal{L} \rho(y)$$

Discrete Bochner approach inspired by Caputo-Dai Pra-Posta.

Bakry-Emery has a "calculus" to describe Ricci curvature on mflds. This is also equiv. to the metric-measure case. But this is not the same for discrete case.

This other condition is called "Gamma-2" condition.

Thm. Let $\mathcal{X}=\{0,1\}^N$, $\mathcal{L}\psi(x) := q\sum_{i=1}^{N} \underbrace{(\psi(x^i)-\psi(x))}_{\delta_i\psi(x)}$.

(Notation $x^i=(x_1,x_2,\dots,x_{i-1},-x_i,\dots,x_N)$.)

$\pi(x)=\underbrace{\frac{1}{2^N}}_{=p}$, $x\in\mathcal{X}$. Then, $\mathrm{Ric}\ge 2q$.

Pf. $$T_1 := \langle\hat\rho\nabla\psi,\nabla\mathcal{L}\psi\rangle = -\frac12 pq^2\sum_{x,i,j}\delta_i\psi(x)[\delta_j\psi(x^i)-\delta_j\psi(x)]\hat\rho(x,x^i)$$

$$= -\frac12 pq^2\sum\delta_i\psi(x)[\underbrace{\delta_i\psi(x^i)}_{\alpha}-\underbrace{\delta_i\psi(x)}_{\beta}]\hat\rho(x,x^i).$$

$$T_2 := \frac12\langle\hat{\mathcal{L}}\rho,(\nabla\psi)^2\rangle$$

$$= \frac14 pq^2\sum(\delta_i\psi(x))^2\cdot[\hat\rho_1(x,x^i)\,\delta_i\rho(x) + \hat\rho_2(x,x^i)\,\delta_i\rho(x^i)]$$

partial derivative w.r.t. first coord.

$$= \frac14 pq^2\sum_x(\delta_i\psi(x))^2[\hat\rho_1(x,x^i)\rho(x^i)+\hat\rho_2(x,x^i)\rho(x^{ii})-\hat\rho(x,x^i)]$$

$x\mapsto x^i$

$$= \frac14 pq^2\sum_i(\delta_i\psi(x^i))^2\underbrace{[\hat\rho_1(x^i,x^{ii})\rho(x)+\hat\rho_2(x^i,x^{ii})\rho(x^i)]}_{A} - \frac14 pq^2\sum(\delta_i\psi(x))^2\hat\rho(x,x^i)$$

$$= \frac14 pq^2\sum\{(\delta_i\psi(x^i))^2-(\delta_i\psi(x))^2\}\hat\rho(x,x^i) + \frac14 pq^2\sum(\delta_i\psi(x^i))^2\underbrace{[A-\hat\rho(x,x^i)]}_{\ge 0}$$

already shows $\mathrm{Ric}\ge 0$.

$$= T_3 + \underbrace{T_4}_{\ge 0}$$

by cancellation of diag. terms in sum. by miraculous prop. of log-mean.

Now $$T_1+T_3 = \frac14 pq^2\sum\hat\rho(x,x^i)\{-2\beta(\alpha-\beta)+\alpha^2-\beta^2\}$$

$$= \frac14 pq^2\sum\hat\rho(x,x^i)\{\delta_i\psi(x^i)-\delta_i\psi(x)\}^2 \ge \frac14 pq^2\sum\hat\rho(x,x^i)\cdot 4(\delta_i\psi(x))^2 = 2q\langle\hat\rho_\psi\nabla\psi,\nabla\psi\rangle$$