E. Milman   Lecture 5   27/02/2015.

1-D Localisation via L^1-OT (B. Klartag '14).

Setup: (M,g) smooth, connected mfld, geodesically convex, $\mu(x) = \Psi(x)\, d\mathrm{Vol}_g(x)$. $\Psi > 0$, $\Psi \in C^2$.

Thm (Klartag). Let $f \in L^1(\mu)$. $\int f\, d\mu = 0$ and.
$\int |f|\, d(x,x_0)\, d\mu(x) < \infty$.

Then $\exists$ Leb-measurable $S \subset M$ s.t.:

① $f|_{M\setminus S} \equiv 0$ μ-a.e.

② $\exists$ f-balanced localisation of $\mu|_S$.

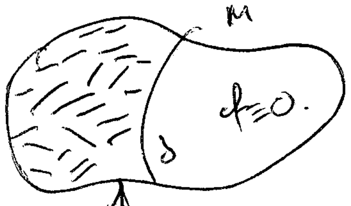
(a) $\mu|_S = \int_\Lambda \mu_\alpha\, d\nu(\alpha)$.

(b) $\forall$ ν-a.e. α, $\int f\, d\mu_\alpha = 0$

(c) 1-d localisation: $\exists\, \Lambda_0 \subset \Lambda$ s.t. $\nu(\Lambda \setminus \Lambda_0) = 0$ s.t.:

(I) $\forall \alpha \in \Lambda_0$, μ_α is a "needle" supported on dist-minimising geodesic $\gamma_\alpha : L_\alpha \subset \mathbb{R} \to M$.
open

(II) $\{\gamma_\alpha\}_{\alpha \in \Lambda_0}$ are disjoint.



(III) $\forall \alpha \in \Lambda_0$, ~~$\mu_\alpha(t_\alpha)\Psi_\alpha(t)\,dt$~~.

$\mu_\alpha = (\gamma_\alpha)_*[\Psi_\alpha(t)\,dt]$

Selection set, needle decomposition

~~Then~~

where $\Psi_\alpha(t) = \Psi(\gamma_\alpha(t)) \cdot J_\alpha(t)$.

with J_α smooth, $J_\alpha > 0$ in L_α.

$-\log \mathrm{Hess}_{n-1}(J_\alpha)(t) \geq$ ~~Ric~~ (like $J_\alpha^{\frac{1}{n-1}}$ concave)

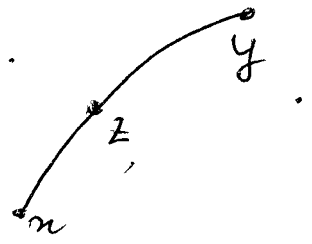
$\geq \mathrm{Ric}_\mu(\gamma_\alpha'(t), \gamma_\alpha'(t))$.

In particular, If $(M^n,g,\mu)\in CD(\rho,N)$ $N\in(-\infty,1)\cup[n,\infty)$.

$\Rightarrow (L_\alpha, |\cdot|, h_\alpha(t)\,dt)\in CD(\rho,N)$.

Def $Z\in$ Strain[u] if $\exists\, x,y\in Z$

s.t. $\begin{cases} u(z)-u(x) = d(x,z) \\ u(y)-u(z) = d(y,z) \\ d(x,y) = d(x,z)+d(z,y) \end{cases}$



"u increases as fast as it can on $x\mapsto y$"

Strain[u] is measurable, and on it, define $\sim$.

$$x\sim y \iff |u(x)-u(y)| = d(x,y).$$

and each equivalence class $\{\gamma_\alpha\}$ is dist minimizing, rel open geodesic.

Thm 1: $\forall$ 1-Lip u, obtain preceding thm w $S=$Strain[u],

ie $\mu|_S = \int_\Lambda \mu_\alpha\, d\nu(\alpha)$. $\forall$ ν-a.e. μ_α spted on $\{\gamma_\alpha\}$.

and $(M^n,g,\mu)\in CD(\rho,N) \Rightarrow (\gamma_\alpha, d, \mu_\alpha)\in CD(\rho,N)$.

Idea of Pf on Strain[u], u diff and $\nabla u(x) = \gamma'_{\alpha(x)}(x)$.
(Feldman-McCann), use Whitney extension thm.

&1
$\exists\, \tilde u\in C^{1,1}(M)$ (via Whitney ext) s.t. $(\tilde u, \nabla\tilde u)$ coincides with $(u,\nabla u)$ on Strain$[u]_\varepsilon$.

For $C^{1,1}$ function, repeat Jacobi calculations, $F_s(x,t) = \exp_x(t\nabla\tilde u(x))$ from $C^{1,1}$ hypersurface.

~~missing~~

Lipschitz. remark, & cts! Not missing stuff! Jessie!

Q. Given f, $\int f \, d\mu = 0$, which u to use in Thm 1. to get 1-d. loc

A. (following Evans–G.?) use maximiser in:

$$\max \left\{ \int f u \, d\mu \; ; \; u \text{ 1-Lip} \right\}.$$

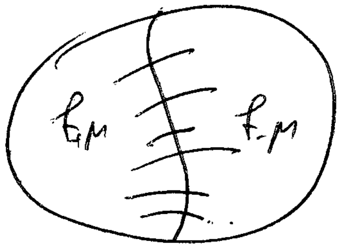
Recall. $$W_1(\nu_1,\nu_2) = \inf_{T_\# \nu_1 = \nu_2} \int d(x, T(x)) \, d\nu_1(x).$$
$$= \sup_{u \text{ 1-Lip}} \int u \, (d\nu_1 - d\nu_2).$$

$$\int f \, d\mu = 0 = \underbrace{\int f_+ \, d\mu}\; - \int f_- \, d\mu.$$

~~have so~~ so f_+, f_- have same mass.

So maximiser u in this problem is Kantorovich pot. for OT-problem transporting $f_+\mu$ to $f_-\mu$.

Strain $[u]$ = OT-rays.



So, in each needle,
$\int f_+ \, d\mu_x = \int f_- \, d\mu_x$ because of mass constraint in OT,
$\Rightarrow \int f \, d\mu_x = 0$.

Rmk. Monge Kantorovich duality is the reason for L^1 OT. Perhaps b/c $\int f \, d\mu = 0$ is an L^1 condition.

Note: OT gives balancing.

L^2-OT:

✓ ① Sharp BM inequ. ((C-M-S,, S, L-V).

✓ ② Sharp Poincaré on $CD(\rho, N)$, $\rho > 0$ [Lott-Villani, Cordero]

✓ ③ on $\mathbb{R}^n$; Sharp isoperimetric; sharp BM, etc.

NOT give you: Ⓐ Sharp log-Sobolev on $CD(\rho, N)$.

$$\int f^2 \, d\mu = 1 \Rightarrow \int f^2 \log f^2 \, d\mu \le \frac{N-1}{N} \frac{2}{\rho} \int |\nabla f|^2 \, d\mu$$

$(N \ge n)$.

L^2 OT misses this.

Ⓑ - Sharp isoperimetric inequalities in (M^n, g, μ).

Ⓒ Sharp Sobolev ineq. on $CD(\rho, N)$.

Now all of this localises! via L^1-OT.