1-D localisation : Milman. 26/02/2015.

Payne–Weinberger 1960
$$\begin{cases} -\Delta g = \lambda_1 g & \text{in } K \\ \partial_n g = 0 & \text{on } \partial K \\ K \text{ convex} \end{cases}$$

- Gromov–V. Milman 80's.
- Kannan–Lovász–Simonovits 93-95.
- Convexity community, Bobkov.

* Restricted to $\mathbb{R}^n$ (S^n).

Th^m (Gromov–V. Milman/K-L-S).

$\mu = \Psi(x)\,dx$ on $\mathbb{R}^n$ probability.

Ψ cts., $f \in L^1(\mu)$, $\int f\,d\mu = 0$.

$\exists$ f-balanced 1-d localisation $\mu = \int \mu_\alpha \, d\nu(\alpha)$ satisfying:

(I) $\int f \, d\mu_\alpha = 0$ $\forall$ ν-a.e. α.

(II) μ_α spted on interval $L_\alpha \subset \mathbb{R}^n$.

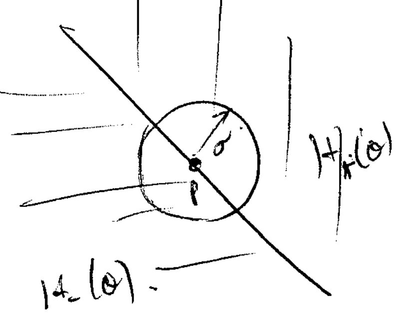
(III). ~~μ_α is supported on interval $L_\alpha \subset \mathbb{R}^n$.~~

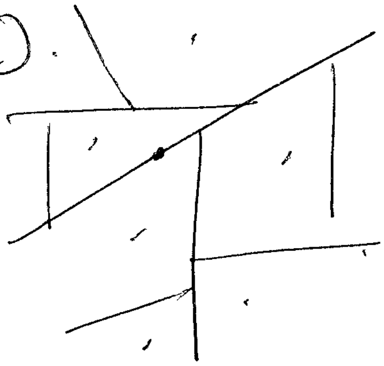
$\mu_\alpha = \Psi(t) J(t)\,dt$ w $J(t)^{\frac{1}{n-1}}$ concave on its convex spt.

$(\mathrm{Spt}(J), |\cdot|, J(t)) \in CD(0,n)$. In particular, if $(\mathbb{R}^n, |\cdot|, \mu) \in CD(\beta, N)$, $N \in (-\infty, 1) \cup [n, \infty]$, then $(L_\alpha, |\cdot|, \mu_\alpha) \in CD(\beta, N)$.

Idea: ① Bisection lemma: $\forall \mu \ll \text{Leb}$, $\int f\,d\mu = 0$, $\forall p \in \mathbb{R}^n$ $\forall S' \subset T_p\mathbb{R}^n$, $\exists \sigma \in S'$ s.t.

$$\int_{H_+(\sigma)} f\,d\mu = \int_{H_-(\sigma)} f.$$





$$\mu = \sum_{\alpha_N \in \{\pm\}^N} \frac{\mu|_{K_{\alpha_N}}}{\mu(K_{\alpha_N})} \mu(K_{\alpha_N}). \qquad \frac{\mu|_{K_{\alpha_N}}}{\mu(K_{\alpha_N})} - \mu_{\alpha_N}.$$

K_{α_N} := intersection of $\frac{1}{2}$ planes convex.

$\int f \, d\mu_{\alpha_N} = 0. \ \forall \alpha_N$.

partition becoming finer as $N \to \infty$, uncountable, so need Rohlin theory + Martingale convergence to obtain.

$\mu = \int \mu_\alpha \, d\nu(\alpha)$, μ_α supported on $K_\alpha = \bigcap_N K_{\alpha_N}$, $\forall \varphi \in L(\mu)$, $\forall \nu$-a.e. α.

$\int \varphi \, d\mu_{\alpha_N} \to \int \varphi \, d\mu_\alpha$ as $N \to \infty$. In particular $\int f d\mu_\alpha = 0$.

Qs. (A) How to ensure K_α 1-d.?

(B) What is the density of μ_α on L_α?

(A) Let $\{A_i\}_{i=1}^\infty$ all $(n-2)$-dim affine subspaces w/ rational coefficients.

$\{x_i = a_i\} \cap \{x_j = a_j\}$ $a_i, a_j \in \mathbb{Q}$.

Given K_{α_N} s.t. $K_{\alpha_{N+1}} = K_{\alpha_N} \cap H_{n+1,\pm}$, where H_{N+1} is the bisecting hyperplane rotating about A_{N+1} (hemisphere)

$\forall \alpha$ $K_\alpha = \bigcap_N K_{\alpha_N}$ is 1-d. Convex is integer dimension.

if $\dim(K_\alpha) \geq 2$; obtain contradiction to construction. $\boxtimes$

(B) <u>Claim</u> if $K_1 \supseteq K_2 \supseteq \dots$ sequence of convex sets, cpct.

$L = \bigcap K_m$ 1-d interval (cpct), then $\mu_m = \frac{\mu|_{K_m}}{\mu(K_m)}$

lechery. w^*-convergent to μ_∞ spted on L,

f.e. $d\mu_\infty = \frac{1}{2}\, 4(e)\, J(t)\, dt$; $J(t)^{\frac{1}{n-1}}$ is concave on its spt.

<u>Brunn (~~taths~~) Concavity principle</u> $\forall K \subset \mathbb{R}^{n+1}$ convex, $\forall \xi \in S^n$

$t \longrightarrow \mathrm{vol}\,(K \cap (t\xi + \xi^\perp))^{\frac{1}{n}}$ is concave on its spt.

<u>Claim</u> Equiv to B-M ineq. in $\mathbb{R}^n$ for A, B, convex:

$\forall A, B \subset \mathbb{R}^n$ convex, $\forall t \in [0,1]$.

$$\mathrm{vol}((1-t)A + tB)^{\frac{1}{n}} \geq (1-t)\,\mathrm{vol}(A)^{\frac{1}{n}} + t\,\mathrm{vol}(B)^{\frac{1}{n}}.$$

Ingredient that used $\mathbb{R}^n$ structure: bisections.

Now, K-L-S sound: $N = (L, \ell(t)^{n-1})$, $\ell(t)$ = affine.

denote $\int_N f\, d\mu = \int_L f(t)\, 4(e)\, \ell(t)^{n-1}\, dt$.

<u>Th^m's</u>. If $\forall N$ extremal $\int_N f\, d\mu = 0 \Rightarrow \int_N g\, d\mu \leq 0$

then $\int_{\mathbb{R}^n} f\, d\mu = 0 \Rightarrow \int_{\mathbb{R}^n} g\, d\mu \leq 0$

<u>Th^m</u>. f_1, f_2 upper semi-cts, f_3, f_4 lower semicts, $\gamma, \beta > 0$. TFAE:

① $\forall K \subset \mathbb{R}^n$ convex. $\left(\int_K f_1\right)^\gamma \left(\int_K f_2\right)^\beta \leq \left(\int_K f_3\right)^\gamma \left(\int_K f_4\right)^\beta$.

② $\forall$ extremal needles $N \subset \mathbb{R}^n$, same inequality $\boxed{\leftarrow \text{1d}}$

1 $\Rightarrow$ 2 by stretching.

2 $\Rightarrow$ 1

$$\mu_\alpha = \underbrace{\chi_K(t)\, \ell(t)^{n-1}}_{\text{extremal preserved by convexity}} dt.$$