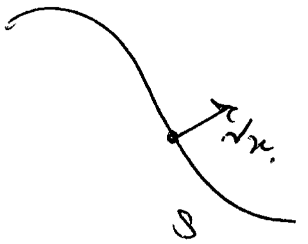
E. Milman. 1D Loc. Lec 3. 25/02/2015.

(M^n, g) smooth, Riem, , $\mu = \Psi\, vol_g$ $\Psi \in C^2$, $\Psi = \exp(-V) > 0$ on M.

S C^2 hypersurface.



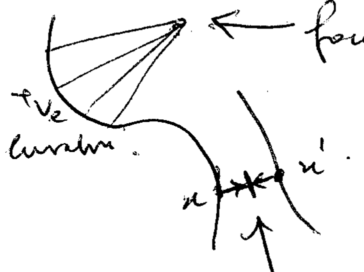
$F_S(x,t) = \exp_x(t v_x)$

$(x,t) \in Dom(F_S(x,t)) \subseteq S \times \mathbb{R}$.

not assuming complete so can't extend geodesics indefinitely.

$L_x \subset \mathbb{R}$ injectivity interval — open interval containing origin s.t. $\forall t \in L_x$, $d(F_S(x,t), S) = |t|$.

L_x usually $\subsetneq \mathbb{R}$, because.



$J_S(x,t)$ — Jacobian of $F_S : (S, vol_S) \times (\mathbb{R}, dt) \to (M, vol_g)$.

- F_S injective on $inj(S) = \{(x,t) \in Dom(F_S) : t \in L_x\}$.
- $vol(Cut(S)) = 0$, where $Cut(S) = Im(F_S) \setminus inj(S)$.
- $J_S > 0$ on $inj(S)$, i.e., F_S diffeomorphism on $inj(S)$.

$$\forall \varphi. \quad \int_{Im(F_S)} \varphi(y)\, \underbrace{dvol_g(y)}_{d\mu} = \int_S \int_{L_x} \varphi(F_S(x,t))\, \underbrace{J_S(x,t)dt}_{J_{S,\mu}}\, \underbrace{vol_S(x)}_{d\mu_S}$$

<u>Can change to:</u> $d\mu$. $J_{S,\mu}$. $d\mu_S$.

$J_{S,\mu} = J_S, \delta_W$, $\quad J_W(x,t) = \dfrac{\Phi(F_S(x,t))}{\Phi(x)}$.

Since true for any test function φ, true for measures, so,

$$\mu|_{\mathrm{Im}(F_S)} = \int_S \mu_x \, d\mathrm{vol}_{S,\mu}(x), \quad \mu_x = J_{S,\mu,x}(t)\,dt, \; F_S(x,\cdot): L_x \to M$$

<u>Th</u> (Generalized Heintze-Karcher.)

Let $(M^n, g, \mu) \in CD(\rho, N)$. $\forall x \in S$, $(L_x, |\cdot|, J_{S,\mu,x}(t)\,dt) \in CD(\rho, N)$

$$-\log \mathrm{Hess}_{N-1} J_{S,\mu,x}(t) \geq \varphi \quad \text{on } L_x.$$

I.e., $CD(\rho, N)$ property inherited by needles.

<u>Pf</u> Note:

$$\frac{A^2}{\alpha} + \frac{B^2}{\beta} \geq \frac{(A+B)^2}{\alpha+\beta}. \quad \forall A, B \in \mathbb{R}.$$

~~Want to use~~

Want to use this with $\alpha = n-1$, $\beta = N-n$.

The only ways $\alpha, \beta > 0$. $(N \in [n, \infty])$ or

$\alpha\beta < 0$ or $\alpha + \beta < 0$ $(N-1 \in (-\infty, 0))$.

This is precisely where the <u>gap</u> comes from.

b/c C.S. is false in $[1, n)$.

Recall Isoperimetric profile. Also, <u>flat Riem profile</u>

$I^b = I^b(\mathbb{R}, |\cdot|)$ only allow $A = (-\infty, \xi]$, or

$A = [\xi, \infty)$.

Gromov-Lévy Program.

- (M^n, g) Riem mfld, smooth, connected, complete. oriented, C^2 bdry ∂M.
- $\mu = e^{-\Psi} \cdot \mathrm{vol}_g$. prob meas.

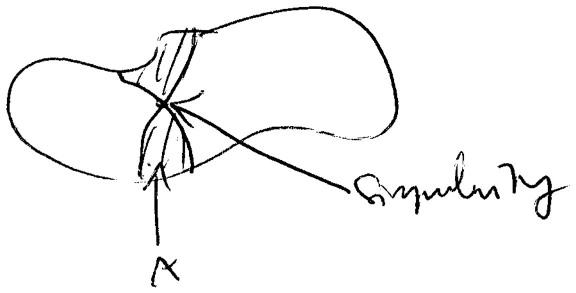
- A given, $\mu(A) = v$ in which $\mu^+(A)$ is minimal $(= I(v))$

Existence, regularity isom minimisers.

(I). $\partial A \cap \mathring{M} = \partial_s A \cup \partial_r A$; $\partial_s A$ - singular ($\mathcal{H}$ dim $\leq n-8$), $\partial_r A$. regular hypersurface. smooth as $e^{-\Psi}$, CMC. $H_{\partial_r A, \mu} \equiv H_\mu(A)$.

(II) Normal rays from. $\partial_r A$ sweep out entire $\mathring{M} \setminus \partial_s A = \mathrm{Im}(F_{\partial_r A})$. (when $\partial M = \emptyset$, Gromov '80).

For ~~dim~~ $n \leq 7$, no singularities!

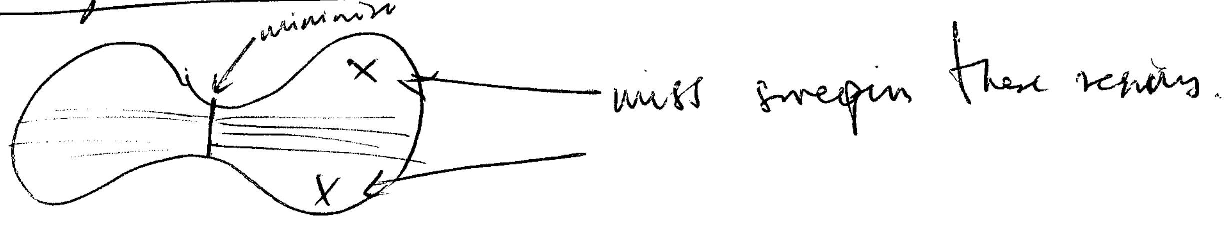


1st variation $\rightarrow$ $\delta^1 \mu^+(A) = H \, \delta^1(\mu(A))$

$\underbrace{J_{s,\mu,x}(0)}$ $\equiv H_\mu(A)$. $\forall x \in \partial_r A$.

Generalised mean curvature, because of weighted measure μ.

<u>Need geodesic convexity.</u>



Conclusion: Every isoperimetric minimizer A induces 1-D localization perpendicular to $S = \partial_r A$.

$$\mu = \int_S \mu_x \, d\mathrm{vol}_{S,\mu}(x).$$

$$\mu_x = J_{S,\mu,x}(t)\, dt \quad \text{on } \exp_x(t v_x) : I_x \to M.$$

which is gen-mean-curv-balanced!

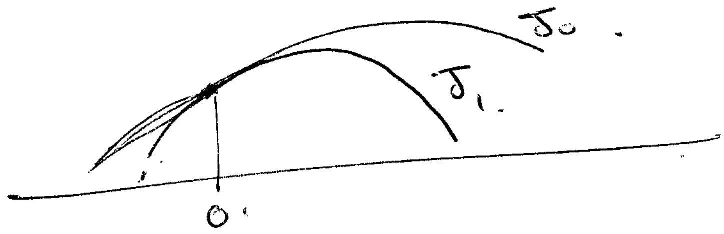
1-D max principle: $N' = N-1$. ← on L_1 certain assum.

$$-\log \mathrm{Hess}_{N'} J_1 = -N' \frac{\left(J_1^{1/N'}\right)''}{J_1^{1/N'}} \geq \rho. \quad J_1(0) = 1, \; J_1'(0) = H.$$

Let J_0 achieve equality in ↑ on max interval. $L_0 \subset \mathbb{R}$ maximal where $J_0 \in (0,\infty)$. Then, $N' > 0$.

$\Rightarrow L_1 \subset L_0$ and $J_1 \leq J_0$ on L_1.

and reverse L_1, L_0 for $N' < 0$.



$\underbrace{J_{H,\rho,N}}_{\text{model } J_0}$ explicitly given → see slides.

1-D max $\Rightarrow$ $J_{\partial_r A,\mu,x} \leq J_{H,\rho,N-1}$. $H = J'_{\partial A,\mu,x}(0)$.

Computation (slides) $\Rightarrow$

$$\mu^+(A) \geq \inf_{H \in \mathbb{R},\, a+b = D = \mathrm{diam}(M)} \max\left(\frac{V}{\int_{-a}^{0} J_{H,\rho,N-1}(t)}, \frac{1-V}{\int_0^b J_{H,\rho,N-1}(t)} \right) =: GL^b_{\rho,N-1,D}(V)$$

↑ Gromov-Lévy Profile.

$V = \mu(A)$ $\quad 1 - V = \mu(M \setminus A)$.

$\underline{\text{Th}^{m}}$. ~~$N\in$~~ (M. '12, '14).

(I) $N\in(-\infty,1)\cup[n,\infty]$, $\forall v\in(0,1)$.

$$I(M^n,g,\mu)(v)\geq GL^{b}_{\rho,N-1,D}(v).$$

(II) If $D=\infty$ or $N\in(-\infty,\underbrace{0]}_{\text{include } 0}\cup[n,\infty)$, then

$$GL^{b}_{\rho,N-1,D}=I^{b}_{\rho,N-1,D}.$$

(I) gives sharpness for $n=1$, $v\in[0,1]$.

But actually, sharp for all $n\geq 2$, $N\in[n,\infty]$, $v\in[0,1]$, ρ, D.

Give complete classification for $CDD(\rho,N,D)$ – Clarkes.