Emmanuel Milman. 1-D loc. lec 2. 24/02/2015.

(M^n, g), geodesically convex; $\mu(x) = \Psi(x)\lambda_g(x)$, $\Psi > 0$ cts.

Assume: $\forall \int f\,d\mu = 0$, $\exists \mu = \int_\Lambda \mu_\alpha\,d\nu(\alpha)$ s.t. $\forall \alpha$ ν-a.e., $\int f\,d\mu_\alpha = 0$.

② Reduction of Poincaré-type ineqs to 1-d:

Want: Find $C_p = C_p(M, g, \mu)$, s.t. $\forall f$ $\int f\,d\mu = 0$,

$$\Rightarrow \int f^2\,d\mu \leq C_p \int |\nabla f|^2\,d\mu.$$

Reduction: given f as above; let $\mu = \int \mu_\alpha\,d\nu(\alpha)$, f-balanced 1-d localisation.

Since $\int f\,d\mu_\alpha = 0$, by 1-d ineq's

$$\int f^2\,d\mu_\alpha \leq C_p^{(\alpha)} \int |\langle \nabla f, \dot\gamma_\alpha(t)\rangle|^2\,d\mu_\alpha(t).$$

$$\leq (\max C_p^\alpha) \cdot \int |\nabla f|^2\,d\mu_\alpha.$$

So, $\int \cdot\,d\nu(\alpha)$, obtain full n-dim. ineq.

(*) Same method would work for Log-Sobolev etc.

③ Reduction of 4-integral inequalities to 1-D:

Given $f_1, \dots f_4 \geq 0$, $\beta, \gamma > 0$, and want to prove

$$\left(\int f_1\,d\mu\right)^\alpha \left(\int f_2\,d\mu\right)^\beta \leq \left(\int f_3\,d\mu\right)^\gamma \left(\int f_4\,d\mu\right)^\beta \quad (4BAL)$$

not saying this is always true, but in some context.

E.g. (I) $\left(\int |f|^p d\mu\right)^{\frac{1}{p}}\left(\int 1 d\mu\right)^{\frac{1}{q}} \le \left(\int c|\nabla f|^q\right)^{\frac{1}{q}}\left(\int 1 d\mu\right)^{\frac{1}{p}}$.

(II) Nash inequality

(III) $\left(\int |f|^p d\mu\right)^{2/p}\left(\int 1 d\mu\right)^{1} \le \left(\int 1 d\mu\right)^{2/p}\left(\int \left(|f|^2 + c|\nabla f|^2\right) d\mu\right)$

(IV). Express heat content in 2-integral inequality.

Reduction: Let $\mu = \int \mu_\alpha \, d\nu(\alpha)$ g-balanced 1-d loc. for $g := f_3 - c f_1$ where $c = \frac{\int f_3 \, d\mu}{\int f_1 \, d\mu} = \frac{\int f_3 \, d\mu_\alpha}{\int f_1 \, d\mu_\alpha}$ $\forall \alpha$.

integrates to 0, so allowed to stick it into black box!

Claim: enough to prove:

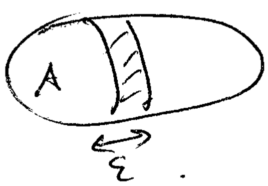
$$\left(\int f_1 \, d\mu_\alpha\right)^{\gamma}\left(\int f_2 \, d\mu_\alpha\right)^{\beta} \le \left(\int f_3 \, d\mu_\alpha\right)^{\gamma}\left(\int f_4 \, d\mu_\alpha\right)^{\beta}.$$

Because $\Leftrightarrow$ $\int f_2 \, d\mu_\alpha \le c^{\beta/\gamma} \int f_4 \, d\mu_\alpha$ $\xrightarrow{\text{integrate}}$

$\int f_2 \, d\mu \le c^{\beta/\gamma} \int f_4 \, d\mu \Leftrightarrow$ (4BAL).

(4) Isoperimetric Ineq.

$$\mu_d^+(A) = \liminf_{\varepsilon \to 0} \frac{\mu(A_\varepsilon^d \setminus A)}{\varepsilon}.$$



$A_\varepsilon^d := \{y \in M : d(y,A) < \varepsilon\}$.

Isoperimetric problem: find Iso. profile $I : \mathbb{R}_+ \to \mathbb{R}_+$ s.t. $\mu_d^+(A) \ge I(\mu(A))$.

Ex. $(\mathbb{R}^n, |\cdot|, \mathrm{vol})$, $\mathrm{Vol}^+_{|\cdot|}(A) \geq n\, \mathrm{vol}(B_2^n)^{\frac{1}{n}}\, \mathrm{vol}(A)^{\frac{n-1}{n}}$.

Ex. Prove this using B-M

$$\mathrm{vol}(A + \varepsilon B_2^n) \geq \left(\mathrm{vol}(A)^{\frac{1}{n}} + \varepsilon\, \mathrm{vol}(B_2^n)^{\frac{1}{n}}\right)^n.$$

$(\mathbb{R}^n, \|\cdot\|, \mathrm{vol})$, $(\mathbb{R}^n, \|\cdot\|, \mu = q(x)\,dx)$ $q^{\frac{1}{N-n}}$ is concave and 1-hom.

Reduction: Given $A \subset M$, let $f = 1_A - \mu(A)$, let $\mu = \int \mu_\alpha \, d\nu(\alpha)$ be 1-d. f-balanced loc.

$$\mu^+_d(A) = \liminf_{\varepsilon \to 0} \int_A \frac{\mu_\alpha(A^d_\varepsilon \setminus A)}{\varepsilon}\, d\nu(\alpha) \geq$$

Fatou

$$\geq \int \liminf_{\varepsilon \to 0} \frac{\mu_\alpha(A^d_\varepsilon \setminus A)}{\varepsilon}\, d\nu(\alpha)$$

$$\geq \int \liminf_{\varepsilon \to 0} \frac{\mu_\alpha((A \cap \gamma_\alpha)_\varepsilon \setminus (A \cap \gamma_\alpha))}{\varepsilon}\, d\nu(\alpha)$$

by def $= \int \mu^+_\alpha(A \cap \gamma_\alpha)\, d\nu(\alpha)$.

If we knew how to solve the 1-d problem

$$\geq \int I(\mu_\alpha(A \cap \gamma_\alpha))\, d\nu(\alpha)$$

$\mu_\alpha(A) = \mu(A)$ α ν-a.e.

f-balanced!

$$= I(\mu(A)) \underbrace{\int d\nu(\alpha)}_{=1}$$

probability measure (assume).

f balanced: $0 = \int f \, d\mu_\alpha = \int (\chi_A - \mu(A))\, d\mu_\alpha$

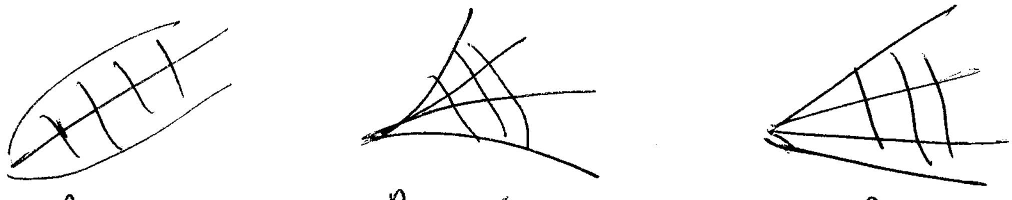
$= \mu_\alpha(A) - \mu(A)$.

I.e., $\mu_\alpha(A) = \mu(A)$ a.e.

Setup (M^n, d) smooth, connected, geodesically convex,

$\mu = q(x)\, d\lambda_g$. $q > 0$, $q \in C^2(M)$, $q = e^{-V}$.

$n-1$ dim. vol. element.



Ric > 0 Ric < 0. Ric = 0.

Def^n (Bakry). N-dim generalised Ric curvature tensor, $N \in (-\infty, \infty]$:

$$\mathrm{Ric}_{g,\mu,N} = \mathrm{Ric}_g - \log \mathrm{Hess}_{N-n} q,$$

$$\log \mathrm{Hess}_a q = \nabla^2 \log q + \frac{1}{a} \nabla \log q \otimes \nabla \log q$$

check, true. $$= a \, \frac{\left(\nabla^2 (q^{\frac{1}{a}})\right)}{q^{\frac{1}{a}}}.$$

Convention: ~~$\infty \cdot 0$~~. $\infty \cdot 0 = 0$.

(a) $N = n$, then $\mathrm{Ric}_{g,\mu,N}(v,v) > -\infty$ $\forall v \in TM$

iff $q \equiv \text{const.}$ and

$\mathrm{Ric}_{g,\mu,n} = \mathrm{Ric}_g$, classical.

(b) $N = \infty$, $\mathrm{Ric}_{g,\mu,\infty} = \mathrm{Ric}_g + \mathrm{Hess}_g V$.

Def (Bakry–Émery): (M^n, g, μ) satisfies $CD(\rho, N)$ if

$$Ric_{g,\mu,N} \geq \rho g \quad \text{on } M.$$

Rem. original B–É defⁿ equivalent if $N \in (-\infty, 0) \cup [n, \infty]$.

Examples

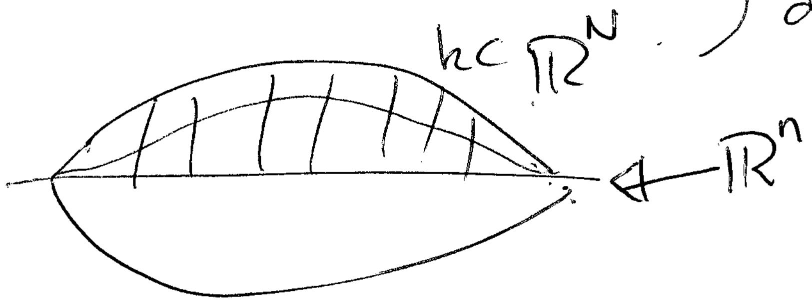
① $K \subset (\mathbb{R}^n, |\cdot|)$ convex, $(K, |\cdot|, \mathcal{L}|_K) \in CD(0,n)$

② $(S^n, g_{round}, Vol_g) \in CD(n-1, n)$.

③ $(\mathbb{R}^n, |\cdot|, \gamma_n = c_n e^{-|x|^2/2} dx) \in CD(1, \infty)$.

④ $(\mathbb{R}^n, |\cdot|, \mu = \psi(x) dx)$, $(N-n)\psi^{\frac{1}{N-n}}(x)$ is concave on convex support, so $\in CD(0, N)$.

dimension remembers where it came from.



⑤ $(\mathbb{R}^n, |\cdot|, \mu = e^{-V})$ V convex. $\in CD(0, \infty)$.

"log concave" densities.

$N \geq 1$ fails b/c $N-1$ slices along lines is the right object. Now, $N \mapsto \frac{1}{N-n}$, gap disappears.

Thm. "integrable needle-decomposition preserves $CD(\rho, N)$" $N \in (-\infty, 1) \cup [n, \infty]$.

⑥ $\left(\mathbb{R}, \frac{1}{\pi(1+|x|^2)}\right)$ or $\left(\mathbb{R}^n, \frac{c_{n,\alpha}}{(1+|x|^2)^{\frac{n+\alpha}{2}}}\right)$ $\alpha > 0$.

$\in CD(0, -\alpha)$. "Cauchy measures".