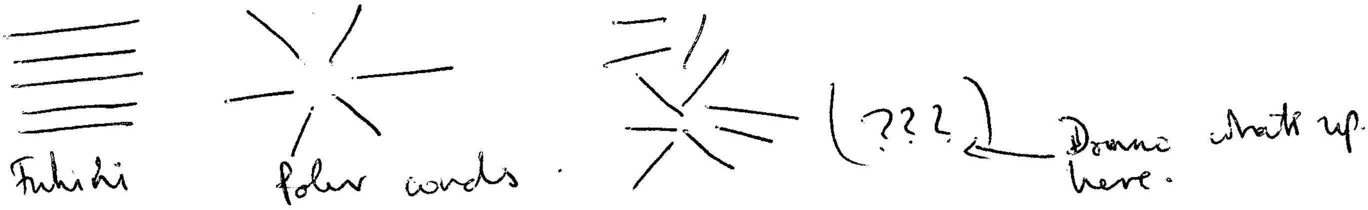
E. Milman. 1-D localisation 23/02/2015.

Goal: Show to prove various analytic/geometric inequalities by reduction to 1-d analysis.



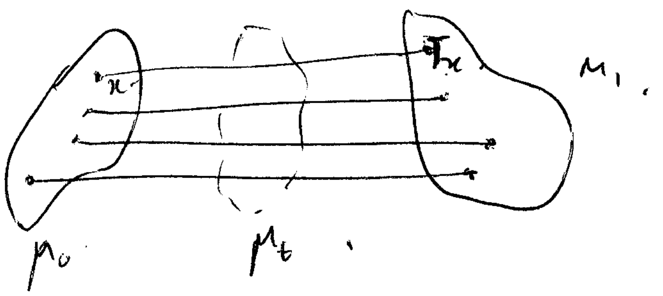
Example from OT: $\mu_1, \mu_2 \ll$ Leb. on $\mathbb{R}^n$, $\int |x|^2 d\mu_i < \infty$. (prob measures).

$$W_2^2(\mu_1,\mu_2) := \min_{T_*(\mu_1)=\mu_2} \int |Tx - x|^2 \, d\mu_1(x)$$

Brenier map T realises this min uniquely. (μ_0 - a.e.).
Characterised by $T = \nabla \varphi$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ convex.

McCann '97: $\mu_t = \underbrace{[(1-t)Id + t\,T]}_{T_t}{}_*(\mu_0)$, $\varphi_t = \frac{d\mu_t}{dx}$,

interpolation along 1-dim transport guide.



μ_t is, in fact, the geodesic b/w μ_0 and μ_1 in W_2.

Def^n: $\mathbb{E}(\varphi_t) := \int u(\varphi_t)\, dx$ is called displacement convex if $[0,1] \ni t \longmapsto \mathbb{E}(\varphi_t)$ is convex $\forall \mu_0, \mu_1 \ll$ Leb.

McCann Characterisation: Enough to check displacement convexity when transforming infinitesimal n-dim vol elements which amounts to checking Jacobian of T_t. "1-d reduction"

↑

Take this as a principle.

Thm (Borell / Brascamp-Lieb '76). Let $\mu = \Psi(x)\,dx$ in $\mathbb{R}^n$,

$N \in (-\infty, 0) \cup [n, \infty]$, TFAE:

(I) $(N-n)\Psi^{\frac{1}{N-n}}$ is concave on its convex supp.

(II) $\forall A, B \subset\subset \mathbb{R}^n$ $\mu((1-t)A + tB) \geq \left[(1-t)\mu(A)^{\frac{1}{N}} + t\mu(B)^{\frac{1}{N}}\right]^N$.

Rem. (I) $A+B = \{a+b : a \in A, b \in B\}$,

(II) $N = n \Rightarrow \Psi \equiv 1$, $\mu = \mathrm{Leb}$ (Brunn-Minkowski).

(III) $N = \infty$, Prékopa-Leindler, $\mu((1-t)A + tB) \geq \mu(A)^{1-t}\mu(B)^t$.

(IV) $N < 0$ ineq. are reversed.

Pf (II) $\Rightarrow$ (I); use $A = x + B(a\varepsilon)$, $B = y + B(b\varepsilon)$, take $\varepsilon \to 0$ and optimize on $a, b > 0$.

(I) $\Rightarrow$ (II): Prove for $N \geq n$. Let $T = \nabla\varphi$ be the Brenier map from $\mu_0 = \Psi_0\mu$ to $\mu_1 = \Psi_1\mu$.

$\mu_t = (T_t)_*(\mu_0)$ $\Psi_t = \frac{d\mu_t}{d\mu}$.

Step 1: Show that claim follows from displacement convexity of $t \mapsto \mathrm{Ent}_N(\mu_t(\mu)) = -\int \Psi_t^{\frac{N-1}{N}}\,d\mu$.

[Hint: $\Psi_0 = \frac{\chi_A}{\mu(A)}$, $\Psi_1 = \frac{\chi_B}{\mu(B)}$].

$(1-t)A + tB \supset \mathrm{spt}(\mu_t)$. Use Jensen's inequality to get.

$$\mu(\mathrm{spt}(\mu_t))^{\frac{1}{N}} \geq -\mathrm{Ent}(\mu_t/\mu).$$

Step 2: Show that $\int \varphi_t(y)^{\frac{N-1}{N}}\, d\mu(y) = \int J_t(x)^{\frac{1}{N}} \rho_0^{\frac{N-1}{N}}\, d\mu$.

$y = Tx$.

Reduces to 1-d analysis, ie, that.

$t \mapsto J_t(x)^N$ is concave.

Step 3: $J_t(x) = J_t^G(x)\, J_t^W(x)$. J_t^G geometric jacobian. J_t^W weighted jacobian.

[$J_t(x)$ is the Jacobian of $T_t(x) = (1-t)x\, t\nabla\varphi$ w.r.t μ].

$$J_t^G(x) = \det\big((1-t)Id + t\nabla^2\varphi\big).$$

$$J_t^W(x) = \frac{\Psi(T_t(x))}{\Psi(x)}.$$

$(J_t^G(x))^{\frac{1}{n}}$ concave b/c $\det^{\frac{1}{n}}$ is concave on pos. def. matrices.

$(J_t^W(x))^{\frac{1}{N-n}}$ concave by assumption — $\Psi^{\frac{1}{n}}$ concave in interval.

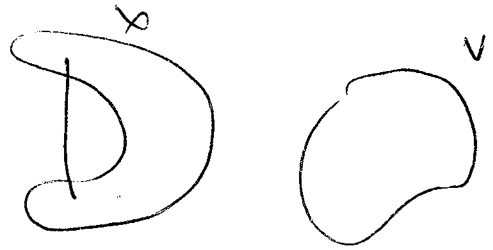
and then by Hölder $J_t(x)^{\frac{1}{N}}$ is concave.

1-D localisation

Setup (M,g) smooth, connected Riem mfld.

induced geodesic distance

$$d(x,y) = \inf\left\{ \int_a^b |\dot\gamma(t)|\, dt \;;\; \gamma(a) = x,\ \gamma(b) = y \right\}.$$

geodesically convex: 

$\forall x, y \in M$, $\exists$ dist minimizing geodesic connecting x and y inside set.

$\mu(x) = \psi(x)\, d\lambda_g(x)$ ← volume.

$\psi > 0$ and continuous, λ_g = volume form $\sqrt{\det g_{ij}}\, dx_1 \dots dx_n$.

<u>Def^n</u>. $\mu = \int_\Lambda \mu_\alpha \, d\nu(\alpha)$ is called a disintegration of μ.

① (Λ, F, ν) complete measure space.

② $\forall$ Leb. measurable $A \subseteq M$

ⓐ $\Lambda \ni \alpha \longmapsto \mu_\alpha(A)$ is defined ν-a.e. and is F-measurable.

ⓑ $\mu(A) = \int_\Lambda \mu_\alpha(A)\, d\nu(\alpha)$.

<u>Def^n 1-d. localization of μ</u>

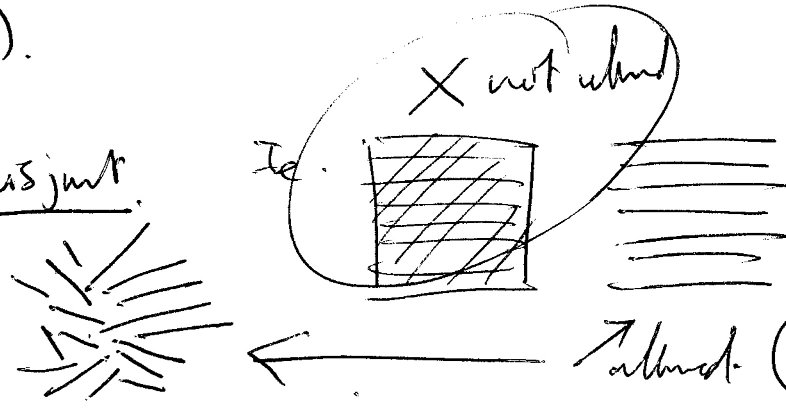
is a disintegration $\mu = \int_\Lambda \mu_\alpha \, d\nu(\alpha)$ s.t.

$\exists \Lambda_0 \subset \Lambda$ which $\nu(\Lambda \setminus \Lambda_0)$ s.t.

① $\forall \alpha \in \Lambda_0$, μ_α is a "1-d" needle, i.e. supported on a distance minimizing geodesic $\gamma_\alpha : L_\alpha \subset \mathbb{R} \to M$.

$\mu_\alpha(A \cap \gamma_\alpha) = \mu_\alpha(A)$.

② $\{\gamma_\alpha\}_{\alpha \in \Lambda_0}$ are <u>disjoint</u>.



Def^{n} Let $f \in L^1(\mu)$ $\int f \, d\mu = 0$.

A 1-d localisation is called f-balanced if $\forall$ ν-a.e. α $\int f \, d\mu_\alpha = 0$.

↗ such a thing is problem dependent, and extremely powerful!

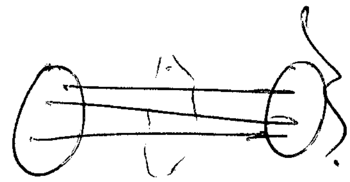
$\boxed{\text{Ex:}}$

(I) Reduction of ... -type ineqs to 1-d:

Want $\mu((1-t)A + tB) \overset{?}{\geq} F(\mu(A), \mu(B), t, \tilde{d}(A,B))$.

$F(v_1, v_2, t, \tilde{d})$ homogeneous in (v_1, v_2) and monotone in $\tilde{d}$.

$(1-t)A + tB = \{ z \subset A : \exists x \in A, \exists y \in B. \ d(x,y) = \frac{d(x,z)}{t} = \frac{d(z,y)}{t}$



Given $A, B \subset M$, set $f = \frac{\chi_A}{\mu(A)} - \frac{\chi_B}{\mu(B)}$. $\int f \, d\mu = 0$.

Let $\mu = \int \mu_\alpha \, d\nu(\alpha)$ f-balanced localisation.

If $\mu_\alpha((1-t)(A \cap \gamma_\alpha) + t(B \cap \gamma_\alpha)) \geq$

$$F(\mu_\alpha(A \cap \gamma_\alpha), \mu_\alpha(B \cap \gamma_\alpha), t, \tilde{d}(A \cap \gamma_\alpha, B \cap \gamma_\alpha))$$

$$\geq \frac{\mu_\alpha(A \cap \gamma_\alpha)}{\mu(A)} F(\mu A ; \mu B, t, d(A,B))$$

↗

$\int f \, d\mu_\alpha = 0$ $\frac{\mu_\alpha(A)}{\mu(A)} = \frac{\mu_\alpha(B)}{\mu(B)}$.

+ homogeneity.

Since $(1-t)A + tB \supset \bigcup_\alpha (1-t)(A \cap \gamma_\alpha) + t(B \cap \gamma_\alpha)$.

and subadditivity of measure.

$$\mu(\quad) \geq \mu(\quad).$$

$$\int \mu_\alpha((1-t)(A \cap \gamma_\alpha) + t(B \cap \gamma_\alpha))\, d\nu(\alpha)$$

$$\geq F(\mu(A), \mu(B), t, \tilde{d}(A,B)) \cdot \underbrace{\int \frac{\mu_\alpha(A)}{\mu(A)}\, d\nu(\alpha)}_{=1}.$$

1-d analysis