C. Sormani Lecture 5. 27/02/2015.

Intrinsic Flat Arzela-Ascoli Theorem.

Review from Wednesday:

$$d_F\big(\underbrace{(X_1,d_1,T_1)}_{M_1},\ \underbrace{(X_2,d_2,T_2)}_{M_2}\big) = \inf_{\substack{Z \text{ complete} \\ \varphi_i : M_i \to Z \text{ dist pres}}} d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2)$$ Achieved!

~~achieved~~

M_1, M_2 m dim, can take Z m+1 sect, or Z Banach.

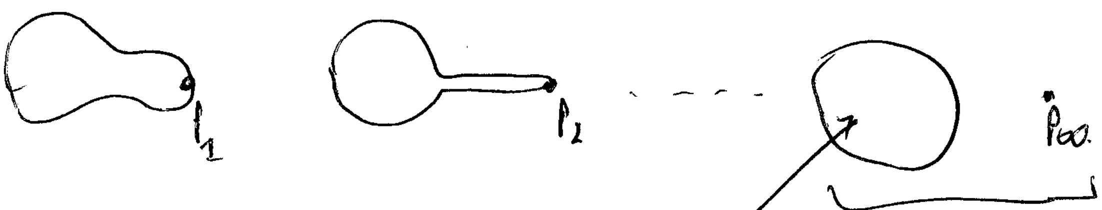
Recall: Th^m $M_i \xrightarrow{F} M_\infty$ precompact $\Rightarrow$ $\exists Z$ comm s.t. $d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0$ (*).

Th^m $M_i \xrightarrow{GH} X$ & $M(M_i) + M(\partial_i) \le C$. then $\exists$ flat convergent subsequence, with $M_\infty \subset X$.

If GH limit is lower dimensional, then F limit is $(0,0,0)$.

Q. When does GH $=$ F?

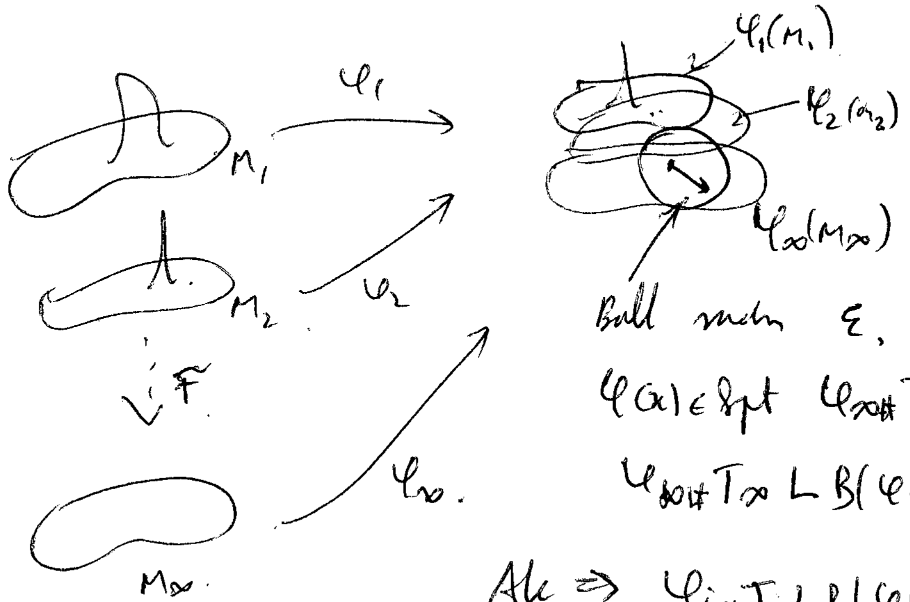
Dissappearing points: Def^n. a sequence $p_i \in X_i$ has no limit in $\bar{X}_\infty$, if when $\varphi_i : X_i \to Z$ satisfying (*), then $\varphi_i(p_i) = z_i \in Z$ converges to $z_\infty \notin \varphi_\infty(\bar{X}_\infty)$.



$\varphi_\infty(\bar{X}_\infty)$ GH limit includes p_∞.

Say: $p_i \in X_i$, $p_i \to p_\infty \in \bar{X}_\infty$ if $\exists \varphi_i : X_i \to Z$ satisfying (*) s.t. $\varphi_i(p_i) \to \varphi_\infty(p_\infty)$.

Lemma. $\forall x \in \overline{X}_\infty$ $\exists x_i \in X_i$ s.t. $x_i \to x_\infty$



Ball radius ε, at $\varphi(x)$.

$\varphi(x) \in \operatorname{spt} \varphi_{\infty\#} T_\infty$,

$\varphi_{\infty\#} T_\infty \llcorner B(\varphi(x), \varepsilon) \neq 0$. ~~for~~ a.e. ε.

$$\text{Also} \Rightarrow \varphi_{i\#} T_i \llcorner B(\varphi(x), \varepsilon) \xrightarrow{wk} \varphi_{\infty\#} T_\infty \llcorner B(\varphi(x), \varepsilon) \neq 0$$

$\Rightarrow$ eventually $\varphi_{i\#} T_i \llcorner B(\varphi(x), \varepsilon) \neq 0$. for large i, and there are the points $x_i \in \operatorname{spt} T_i$.

Lem. If $x, y \in \overline{X}_\infty$, $\exists x_i, y_i \in X_i$ s.t. $x_i \to x$, $y_i \to y$ and $d_i(x_i, y_i) \to d_\infty(x, y)$.

[So, $\liminf_{i\to\infty} \operatorname{diam}(X_i) \geq \operatorname{diam}(X_\infty)$] h

Diam can drop!, ~~Are the~~ because $p_\infty \notin \varphi_\infty(\overline{X}_\infty)$. as from before.

Lem. p_i has no limit in $\overline{X}_\infty$, then $\exists R > 0$ s.t. $r < R$

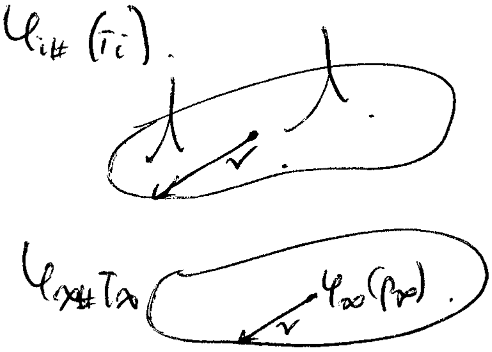
$d_F(S(p_i, r), 0) \to 0$. where $S(p_i, r) = (\overline{B}(p_i, r), d_i, T_i \llcorner \overline{B}(p_i, r))$

balls viewed as integral current spaces.

__Lemma.__ Replace argmin in first lemma with $=0$,
then set: $\varphi_{i\#} T_i \llcorner B(z_\infty, \varepsilon) \longrightarrow 0$.
for $z_\infty \notin \varphi_\infty(\bar{X}_\infty)$.

__Lemma.__ $p_i \to p_\infty \in \bar{X}_\infty$ $\exists$ subseq. s.t. a.e. in r,
$S(p_i, r) \xrightarrow{\mathcal{F}} S(p_\infty, r)$.

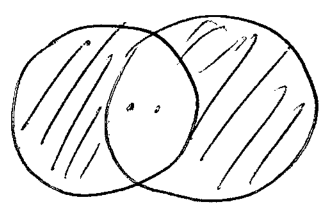
__Pf idea:__ $\varphi_{i\#}(T_i)$.



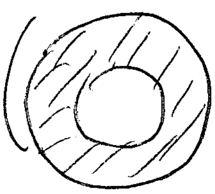
$d_Z(\varphi_i(p_i), \varphi_\infty(p_i))$ small.

$$d_{\mathcal{F}}^{Z}\Big(\varphi_{j\#} T_j \llcorner \underbrace{B(\varphi_\infty(p_\infty), r)}_{\text{But need this to be } B(\varphi_j(p_j), r).},\ \varphi_{\infty\#} T_\infty \llcorner B(\varphi_\infty(p_\infty), r)\Big).$$

$\mathbf{M}\Big($



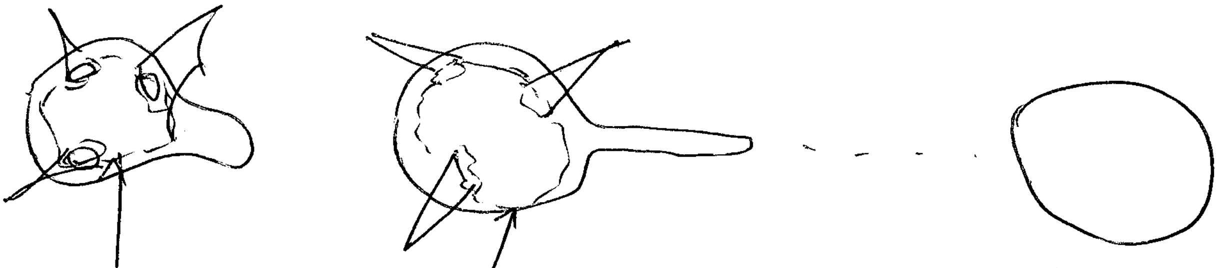
$\Big) \le \mathbf{M}\Big($



$\Big)$.

But impossible to control this, so need subsequence.

__Thm__ ($\mathcal{F}$ to GH).
If $M_i^m \xrightarrow{\mathcal{F}} M_\infty^m$ precompact $\neq 0$.
$\exists$ S_i integral current in $\bar{X}_i$ s.t. $N_i = (\text{set } S_i, d_i, S_i) \subset M_i$
$N_i \xrightarrow{GH} \bar{M}_\infty$ and $\liminf_{i \to \infty} \mathbf{M}(N_i) \geq \mathbf{M}(M_\infty)$.



N_1 N_2. $N_\infty = M_\infty$.

Arzela-Ascoli Th$^{\underline{m}}$.

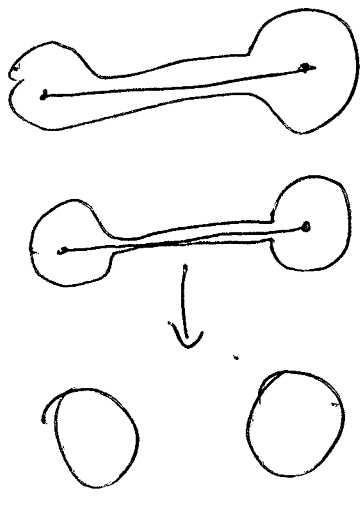
Fix $\lambda > 0$, $M_i^m \xrightarrow{\mathcal{F}} M_\infty^m$ precompact, $F_i : M_i^n \to W$.
W compact metric space, $\mathrm{Lip}(F_i) \le \lambda$.
Then $\exists$ subsequence $F_{i_j} \to F_\infty$; $F_\infty : M_\infty^n \to W$, $\mathrm{Lip}(F_\infty) \le \lambda$.
If $p_i \to p_\infty$ then $F_i(p_i) \to F_\infty(p_\infty)$.

Contrast w/ GH Arzela-Ascoli:

$F_i : M_i \to W_i$, $M_i \xrightarrow{GH} M_\infty$, $W_i \xrightarrow{GH} W_\infty \leftarrow$ compact.

$\mathrm{Lip}(F_i) \le \lambda \Rightarrow \exists F_\infty : M_\infty \to W_\infty$.

Ex. When in $\mathcal{F}$, we cannot take $W_i \xrightarrow{\mathcal{F}} W_\infty$.] As in, target space fixed, does not change like in GH AA.
even if $M_i = [0,1]$.



W_1. W_2.

Open: $W_i \xrightarrow{GH} W_\infty$ might give F_∞??

Local Isom. Arz Asc:

$$\begin{tikzcd} M_i^m \arrow[r, "F_i"] \arrow[d, "\tilde{F}"'] \arrow[dr, dotted] & W_i^m \arrow[d, "\hat{F}"] \\ M_\infty^m & W_\infty^m \end{tikzcd}$$

F_i are isometries on balls of fixed radius r.

Then, $\exists\ F_\infty : M_\infty \to W_\infty$, a local isometry.

Applied in work of Zalura Lunaei to study $\mathcal{F}$ limits of warping spaces.

Bolzano - Weierstrass Thm:

Old GH version: $M_i \xrightarrow{GH} M_\infty$.

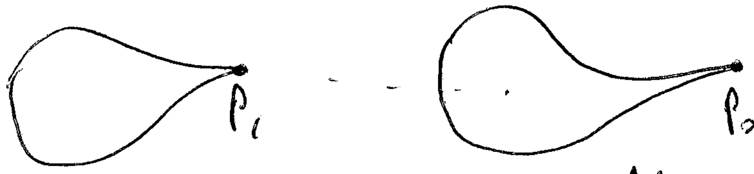
If $p_i \in M_i$ (because Gromov reckons they embed into common compact space), $\exists$ subsequence converging to p_∞.

In Flat: not true; (I) Z is not ~~complete~~ compact.
(II) points can disappear, even if $\varphi_i(p_i)$ have a limit.

Thm. If $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $p_i \in M_i$, and $\exists\ r_0 > 0$ and $h : (0, r_0] \to (0, r_0]$ s.t. $d_{\mathcal{F}}(S(p_i, r), 0) > h(r)$ a.e. r,
then $\exists$ subsequence $p_{i_j} \to p_\infty \in \overline{M_\infty}$.

uniform, positive, no continuity required.

To guarantee $p_\infty \in M_\infty$, need stronger hyp.
Ie, cusp fits hyp.



$p_\infty \in \bar{M}_\infty$

$M_\infty = \bar{M}_\infty \setminus \{p_\infty\}$.

Th^m's where $p_\infty \in M_\infty$ appear in joint work with S.-Portegies

Need: Gromov-Filling.

To prove th^m's guaranteeing GH and F agree.

- Either via Gromov-Filling
- Use BW. Th^m when
  $M_i \xrightarrow{F} M_\infty$, $M_i \xrightarrow{} M_\infty$ GH.

  Take $x \in X_\infty$, $\exists x_i \in M_i$, $x_i \to x_\infty$.

  Show $S(x_i, r)$ are nice enough so that x_i have a limit in $\bar{M}_\infty$.

Th^m w/Wenger: $\partial M_i = \emptyset$ Riem., M_i^n w/ Ricci ≥ 0,
$\mathrm{vol}(M_i) \geq V_0 > 0 \Rightarrow M_\infty = X_\infty$

Th^m Li-Perales: $\partial M_i = \emptyset$, integral currents, $\mathrm{vol}(M_i) \geq V_0 > 0$
spaces with weight 1, satisfy Alex Curv ≥ 0, $M_\infty = X_\infty$.

Perales (soon to appear) Riem w/bdy.