C. Sormani Lecture 3 IFC 25/02/2015

Review of Ambrosio-Kirchheim.

- Current in metric space Z.
- $T(f, \pi_1, \dots, \pi_m)$.
- Integer Rect current.

$$T(f, \pi_1, \dots, \pi_m) = \sum \theta_i \int_{A_i} (f \circ \varphi_i)\, d(\pi_1 \circ \varphi_i) \wedge \dots \wedge d(\pi_m \circ \varphi_i)$$

$\theta_i \in \mathbb{Z}^+$, φ_i bilip, $\varphi_i : A_i \subset \mathbb{R}^n \to Z$ Borel.

- $\partial T(f, \pi_1, \dots, \pi_m) = T(d(f, \pi_1, \dots, \pi_m))$
$= T(1, f, \pi_1, \dots, \pi_m)$.

- Integral Current: T is int rect + ∂T int rec.
- Thm: Need only check $M(\partial T) < \infty$.

- Not mention last time

$$T \llcorner h(f, \pi_1, \dots, \pi_m) = T(f \cdot h, \pi_1, \dots, \pi_m).$$

Also, we were able to extend T acting on $f \in L^1$
so, h can be bad, and so, for some $H \subset Z$, Borel.

$$T \llcorner H(f, \pi_1, \dots, \pi_m) = T(f \cdot \chi_H, \pi_1, \dots, \pi_m).$$

By Ambrosio-Kirchheim Slicing Thm,
$\Rightarrow T \llcorner B(p, r)$ is an integral current for a.e. r.
a.e. r because intersection of bdry with some r can have infinite mass!

Joint w/ Stefan Wenger [SDG].

m-Integral Current Space (X, d, T). metric space (X, d) with an integral current T in $\bar{X}$ s.t. $X = \text{set}(T) = \{x : \liminf_{r \to 0} \frac{\|T\|(B(x,r))}{r^m} > 0\}$

$\|T\|(B(x,r)) = M(T \llcorner B(x,r))$.

<u>Example</u>: M oriented Riem, $M = (X, d, T)$,

$T\omega = \int_M \omega$.

<u>Def</u>. $d_{\mathcal{F}}\big((X_1, d_1, T_1), (X_2, d_2, T_2)\big)$

$= \inf_{\substack{Z \text{ complete} \\ \varphi_i : X_i \to Z \text{ dist pres.}}} \left\{ d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2) \right\}$.

$d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2) = \inf\{ \mathbb{M}(A) + \mathbb{M}(B) : A + \partial B = \varphi_{1\#}T_1 - \varphi_{2\#}T_2 \}$

A is an m int. current

B is an $m+1$ int. current.

<u>Thm</u>. M_1, M_2 precpt, $d_{\mathcal{F}}(M_1, M_2) = 0$ iff $\exists$.

$\exists$ current preserving isometry $\psi : X_1 \to X_2$ s.t.

$\psi_\# T_1 = T_2$. orientation preserving.

$(M_i = (X_i, d_i, T_i))$

<u>Thm</u>. ~~For~~ M_1, M_2 precompact, infimum achieved by ~~an~~ some Z and an A and B, so let $Z' = \operatorname{spt} A \cup \operatorname{spt} B \subset Z$.

Thus the inf in def of d_F can be over metric ~~spaces~~ completions of $(m+1)$ countably $\mathcal{H}^{m+1}$ rectifiable metric spaces. (separable Z)

By Kuratowski Embedding, the inf can be over Banach spaces Z.

$\downarrow$ $Z \hookrightarrow l^\infty(Z)$ Schilo $x \mapsto d(x, \cdot)$.

Thm. If M_i are oriented Lipschitz mflds,
(X_i, d_i, T_i) $\quad T_i\omega = \int_{M_i}\omega$.

$$d_{\mathcal F}(M_1, M_2) \le \tfrac{1}{2}(m+1)\,\lambda^{m+1}(\lambda - 1)\left[\mathrm{vol}(M_i) + \mathrm{vol}(\partial M_i)\right].$$

where $\lambda = e^{d_{Lip}(M_1, M_2)}$. $\quad (m = \max\{\dim M_1, \dim M_2\})$.

$$d_{Lip}(M_1, M_2) = \inf_{\substack{\text{bi-Lip}\\ \psi: M_1 \to M_2}} \left\{ \log(\mathrm{Lip}\,\psi) + \log(\mathrm{Lip}\,\psi^{-1}) \right\}.$$

Gromov "Metric Structures".

Joint w/ Lakzian, constructive pf.

Gromov. If $(X_i, d_i) \xrightarrow{GH} (X_\infty, d_\infty)$ all compact,
then $\exists$ common cpct metric space
Z and dist preserving $\varphi_i : X_i \to Z$.
s.t. $\quad d_H(\varphi_i(X_i), \varphi_\infty(X_\infty)) \to 0$.

This allows us to define $p_i \to p_\infty$ for $p_i \in X_i$.
by saying $\varphi_i(p_i) \to \varphi_\infty(p_\infty)$ (up to a certain amount of isometry).

If $p_\infty \in X_\infty$ $\exists\, p_i \in X_i$ s.t. $p_i \to p_\infty$. via Bolzano-Weierstrass;
but this fails in IF. setting b/c of no compact Z.

But: If $M_i \xrightarrow{\mathcal F} M_\infty$ all precompact, $\exists$ common, complete, separable
metric space; countably $\mathcal H^{m+1}$ rectifiable metric space Z. and
$\varphi_i : M_i \to Z$. s.t.
$$d_{\mathcal F}^{Z}(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \to 0. \qquad \boxed{Z \text{ NOT CPCT}}$$

The non compactness is always in the intrinsic case b/c if it were, then $\xrightarrow{F}$ is $\xrightarrow{GH}$ and so nothing new!

By Kuratowski, Z can be Banach!

<u>Wenger</u>: All currents in B-spaces,

$$d_F^Z(T_j, T_\infty) \to 0 \quad \text{iff} \quad T_j \to T_\infty.$$

assuming that $\mathbb{M}(T_j) < C$ + $\mathbb{M}(\partial T_j) < C$.

~~Pullback by Kuratowski to get in common Z.~~

Make Z Ban common Z Banach whenever you need to use this.

<u>Consequences</u> w/Wenger JDG.

lower semicontinuity of mass (from AK, low semi-mass)

If $(X_i, d_i, T_i) \xrightarrow{F} (X_\infty, d_\infty, T_\infty)$ and $\mathbb{M}(T_i) + \mathbb{M}(\partial_i) < C$.

then $\liminf_{i\to\infty} \mathbb{M}(T_i) \geq \mathbb{M}(T_\infty)$.

In fact, view $B(p_i, r)$ as an ICS, (for a.e. r)

$$\liminf_{i\to\infty} \mathbb{M}(B(p_i, r)) \geq \mathbb{M}(B(p_\infty, r)).$$

if $p_i \to p_\infty$.

• If $M_i \xrightarrow{\mathcal{F}} M_\infty$, then $\partial M_i \xrightarrow{\mathcal{F}} \partial M_\infty$.

and get $\liminf_{i\to\infty} \mathbb{M}(\partial M_i) \geq \mathbb{M}(\partial M_\infty)$.

and $\liminf_{i\to\infty} \mathbb{M}(\partial B(p_i, r)) \geq \mathbb{M}(\partial B(p_\infty, r))$.

Rmk. No $\mathbb{M}(\partial_i) \leq C$ needed here!

$\lim_{i\to\infty} \text{Fillvol}_\infty(\partial B(p_i, r)) = \text{Fillvol}_\infty(\partial B(p_\infty, r))$.

in CVPDE w/Wenger.

c.f. arxiv w/ Portegies.

Back to Wenger JDG.

Thm [Wenger & Ambrosio-Kirchheim].

If $(\bar{X}_i, d_i) \xrightarrow{GH} \underbrace{(\bar{X}_\infty, d_\infty)}_{\text{not necessarily IFS}}$. $X_i = \text{set}(T_i)$. $\underline{\underline{i < \infty}}$.

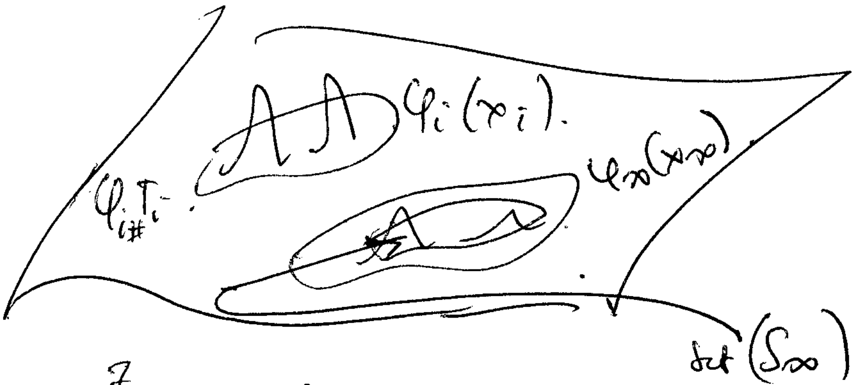
$\mathbb{M}(T_i) + \mathbb{M}(\partial T_i) \leq C$.

then, $\exists$ subsequence.

$(X_{i_n}, d_{i_n}, T_{i_n}) \xrightarrow{\mathcal{F}} (Y_\infty, d_\infty, T_\infty)$.

where $Y_\infty \subseteq X_\infty$ and d_∞ restricted.

Pf. Gromov: $\exists\, Z$ cpt.



$$d_H^Z(\varphi_i(X_i), \varphi_\infty(X_\infty)) \to 0.$$

pushforward $\varphi_{i\#}T_i$ currents in Z.

$$\mathbf{M}(\varphi_{i\#}T_i) = \mathbf{M}(T_i) \le C.$$

Similar to today.

AK cpctness $\to$ subsequence $\varphi_{i_n\#}(T_{i_n}) \to S_\infty$.

set $S_\infty \subset \varphi_\infty(S_\infty)$. $\leftarrow$ not hard.

then Kuratowski, move to $l^\infty(Z)$.

$$(\iota\circ\varphi_{i_n})_\# T_{i_n} \to \iota_\# S_\infty.$$

$$d_F^{l^\infty(Z)}((\iota\circ\varphi_{i_n})_\# T_{i_n}, \iota_\# S) \to 0.$$

$\Downarrow$

$$d_F((X_i, d_i, T_i), (\mathrm{set}\, S_\infty, d_Z, S_\infty)) \to 0.$$