C. Sormani — Ambrosio-Kirchheim's — 24/02/2015.

Lec 2 — "Currents on Metric Spaces".

Recall: Class of spaces: Integral Current Spaces.

(X, d, T) where (X,d) is a metric space.

T is an integral current on $\bar{X}$ s.t. $X = \mathrm{set}(T)$.

T encodes orientation and weight on X.

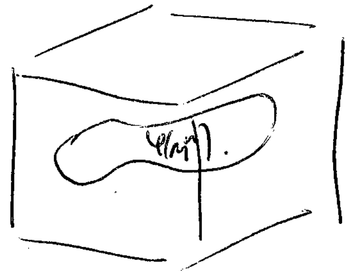
countably $\mathcal{H}^m$-rectifiable → so they have biLipschitz charts, but shouldn't use them

→ Rather properties are based on T rather than charts.

Open question: What about Integral Current Spaces satisfying $CD(K,N)$ or $RCD(K,N)$?
(w.r.t $\mu = \|T\|$ mass measure).

• Historical Review Federer-Fleming (Annals 1960).

submanifolds (oriented). $\varphi : M^m \longrightarrow \mathbb{R}^N$



$\mathbb{R}^N$

$\omega \in \Omega^m(\mathbb{R}^N)$, m-diff form,

$$T(\omega) = \varphi_\#[[M]]\omega = \int_M \varphi^*\omega.$$

If $\omega = f\, d\pi_1 \wedge \dots \wedge d\pi_m$, then

$$\varphi_\#[[M]]\omega = \int_M f\circ\varphi \; d(\pi_1\circ\varphi)\wedge \dots \wedge d(\pi_m \circ \varphi).$$

well defined for Lipschitz φ.

$T_j \to T$ weakly as currents if $T_j\omega \to T\omega$
for all $\omega \in C_c^\infty(\Omega^m(\mathbb{R}^N))$.

Limits of submanifolds of interest for F.F:

Rectifiable currents (integer).

$$T(\omega) = \sum_{i=1}^{\infty} \theta_i \int_{A_i} \varphi_i^* \omega,$$

$A_i \subseteq \mathbb{R}^m$ Borel, $\varphi_i : A_i \to \mathbb{R}^N$ Lipschitz, $\underline{\underline{\theta_i}}$ Integer currents.

Mass: $\mathbb{M}(T) = \sum_{i=1}^{\infty} |\theta_i| \mathcal{H}^m |\varphi_i(A_i)|$ if $\underbrace{\varphi_i(A_i) \cap \varphi_j(A_j) = \emptyset}$

more complicated if not b/c of cancellation.

Boundary: $\partial T(\omega) = T(d\omega)$ $\quad (\varphi_\# T)\omega = T(\varphi^* \omega)$.

$\omega \in C_c^\infty(\mathbb{R}^{N-1})$

Integral current: An integer rectifiable current whose bdry is also integer rectifiable.

Theorem: If T is integer rectifiable and $\mathbb{M}(\partial T) < \infty$, then T is integral.

Thm: $\mathbb{M}(T_i) + \mathbb{M}(\partial T_i) \leq C < \infty$, spt $T_i \subset K$ compact, then a subsequence converges weakly.

Compactness Thm: The limit is also integral current.

<u>Thm</u> Flat convergence and weak convergence agree, with these hypotheses.

For Plateau problem, crucial thing: $T_i \to T$ weakly, then $\liminf_{i\to\infty} \mathbb{M}(T_i) \geq \mathbb{M}(T)$.

- <u>Ambrosio - Kirchheim.</u>

Currents on <u>Complete</u> metric spaces.

What is a diff. form?

De Giorgi $m+1$ tuples; $(f, \pi_1, \dots, \pi_m)$, $f \in \mathrm{Lip}_b$, $\pi_i \in \mathrm{Lip}$.

bdd Lipschitz.

Space $\mathcal{D}^{m+1}$.

$$d\omega = (1, f, \pi_1, \dots, \pi_m).$$

WARNING: $d^2 \neq 0$, and these are <u>not</u> alternating.

$$\varphi^* \omega = (f\circ\varphi, \pi_1\circ\varphi, \dots, \pi_m\circ\varphi).$$ defined for $\varphi: Z \to W$ Lipschitz.

Metric functional: $T: \mathcal{D}^{m+1} \to \mathbb{R}$,

$$T(x+y) \leq T(x) + T(y).$$

$$T(tx) \leq tT(x). \qquad t \geq 0.$$

$$\partial T(\omega) = T(d\omega) \qquad \varphi_\# T\omega = T(\varphi^*\omega).$$

"Finite Mass" if

$$\boxed{|T(f, \pi_1, \dots, \pi_m)| \leq \prod_{i=1}^{\infty} \mathrm{Lip}(\pi_i) \int_Z f \, d\mu.} \quad *$$

for some μ-finite and Borel on Z.

<u>Mass Measure</u>: $\|T\|(A) = \mu(A)$. where μ is the smallest measure satisfying

Mass $\mathbb{M}(T) = \|T\|(Z)$. where Z is the whole space.

$\|\varphi_\# T\| \leq (\mathrm{Lip}\,\varphi)^m \varphi_\# \|T\|$.

<u>Def^n</u>. T is an m-current if

(I). Multilinear in $(f, \pi_1, \dots, \pi_k)$.

(II). $\lim_{i\to\infty} T(f, \pi_1^i, \dots, \pi_k^i) = T(f, \pi_1, \dots, \pi_k)$ if

$\pi_j^i \to \pi_j$ ~~Lip~~ pointwise in Z.

$\mathrm{Lip}(\pi_j^i) \leq C$.

(Note f fixed).

(III). $T(f, \pi_1, \dots, \pi_k) = 0$ if $\exists\, \pi_i$ s.t. $\pi_i \equiv$ const on nhd $\{f \neq 0\}$.

<u>Properties</u>: • product rule; chain rule, continuity where $f^i \to f$ in $L^1(Z, \|T\|)$, stronger locality

$f, \pi_1 \in \mathrm{Lip}_b$.

Product: $T(1, f\pi_1, \pi_2, \dots, \pi_m) = T(f, \pi_1, \pi_2, \dots, \pi_m) + T(\pi_1, f, \pi_2, \dots, \pi_m)$.

Chain: $T(f, \psi\circ\pi_1, \dots, \psi\circ\pi_k) = T(f \det \nabla\psi, \pi_1, \dots, \pi_k)$ if $\psi \in C^1(\mathbb{R}^m, \mathbb{R}^m)$. (at the level of the tuple!)

As a consequence, alternating prop:

$$T(f, \pi_{\sigma(1)}, \dots, \pi_{\sigma(m)}) = (-1)^{\mathrm{sign}(\sigma)} T(f, \pi_1, \dots, \pi_m).$$

All these properties are <u>through</u> the current. Not

Intrinsic properties of the underlying space Z^{m+1}.

Def^n. T is an integer rectifiable current if ______ Complicated.

But following theorem is helpful:

Th^m. T is ~~an~~ integer rectifiable if $\exists\, \varphi_i : A_i \to Z$ bi-Lipschitz, $A_i \subset \mathbb{R}^m$ Borel, $\theta_i : A_i \to \mathbb{Z}_+$ are Borel, s.t.

$$T(f, \pi_1, \dots, \pi_m) = \sum_{i=1}^{\infty} \theta_i \int_{A_i} \underbrace{f\circ\varphi_i \, d(\pi_1\circ\varphi_i)\wedge \dots \wedge d(\pi_m\circ\varphi_i)}_{\text{defined } \mathcal{L}^m\text{-a.e.}} \; d\mathcal{L}^m$$

(NOTE: $\pi_i : Z \to \mathbb{R}$, $\varphi_i : A_i \to Z$, $\pi_i\circ\varphi_i : A_i \subset \mathbb{R}^m \to \mathbb{R}$).

Th^m $\|T\| = \theta\, \lambda\, \mathcal{H}^m \llcorner \operatorname{set}(T)$, $\operatorname{set} T = \{x \in Z : \liminf_{r\to 0} \frac{\|T\|(B(x,r))}{r^m} > 0\}$

Major Th^m.

In F.F., no λ. (Here, $\lambda_x := \frac{2^m}{\omega} \sup\left\{ \frac{\mathcal{H}^m(B(x))}{\mathcal{H}^m(R)} : R \text{ parallelepiped contains } B_1 \right\}$

in approx tangent plane to set T.

~~R is Tangent plane~~.

"Area factor", due to Finsler effect → some tangent planes may be Euclidean, others may not.

Also get ?

Integral currents: Integer rec. + ∂T int rec.

T int rect, $\mathbb{M}(\partial T) < \infty$, T integral.

Compactness, but Wenger proves Flat ⟺ Weak.