C. Sormani 23/02/2014.

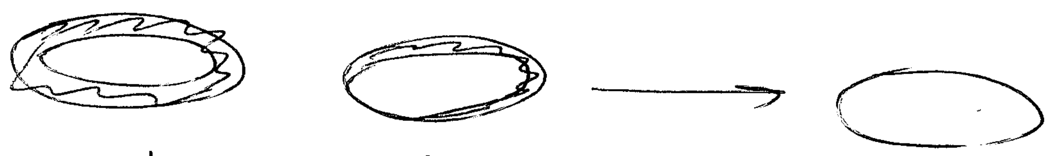
$C^{1,\alpha}$ Convergence: limits are $C^{1,\alpha}$ diff to subsequence.

Cheeger Gromov: $|R| \le \Lambda$, vol $\ge V_0$, diam $\le D$.
Anderson Cheeger: $|Ric| \le \Lambda$, inj $\ge i_0$, diam $\le D$.

Gromov-Hausdorff Conv:
limits are compact geodesic metric spaces.
Topology and dimension can change in limit

Intrinsic Flat Convergence
limits are rectifiable, oriented, $\mathcal{H}^n$ rectifiable spaces.
Wenger: diam $\le D$, vol $\le V$.

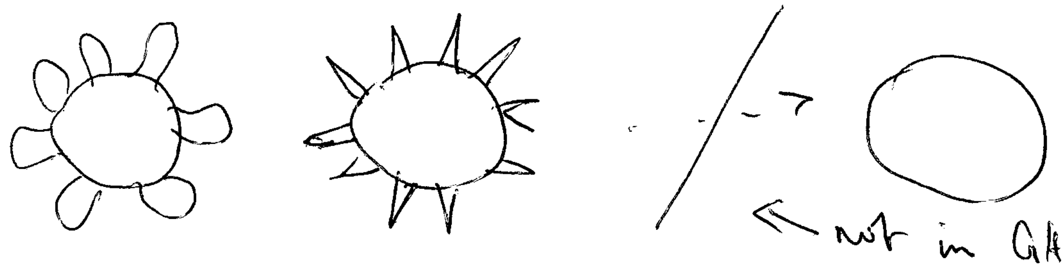
GH: $d_H^Z(A,B) = \inf\{$ ~~d(A,B)~~ ——— $\}$.
$d(A,B) = \inf\{$ ——— all embeddings $\}$.



* Upper bound → find a simple metric space.

Gromov's compactness thm.

Flat Convergence motivation: Ilmanen asked (for GR).



← not in GR.

• GH Convergence does <u>not</u> preserve rectifiability.

• Recall Flat convergence. $M^n \subset \mathbb{R}^{n+k}$,

$$F_M : \mathbb{R}\, C_c^\infty(\Omega^n(\mathbb{R}^{n+k})) \to \mathbb{R}.$$

$$F(M, (\varphi)) = \int_M \varphi \, d\mathcal{H}^n :=$$

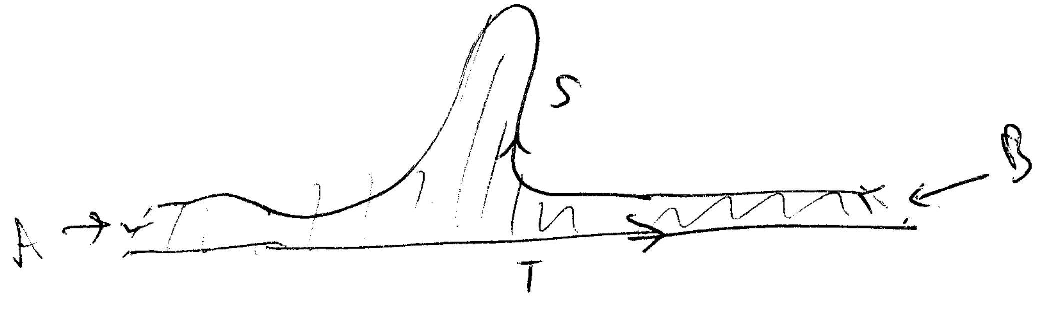
Does $F(M_i, \varphi) \xrightarrow{?} F(M_\infty, \varphi) \quad \forall \varphi \in C_c^\infty(\Omega^n)$.

• Intrinsic Flat: distance from $\mathbb{R}^{n+k}$, just like Hausdorff to G.H.

• Flat distance

$d_F(T, S) := \inf \{ M(A) + M(B) : A + \partial B = S - T \}$

$$F_{\partial B}(\varphi) := F_B(d\varphi), \quad M(A) = \text{mass of } A.$$



GH distance (see's splines).

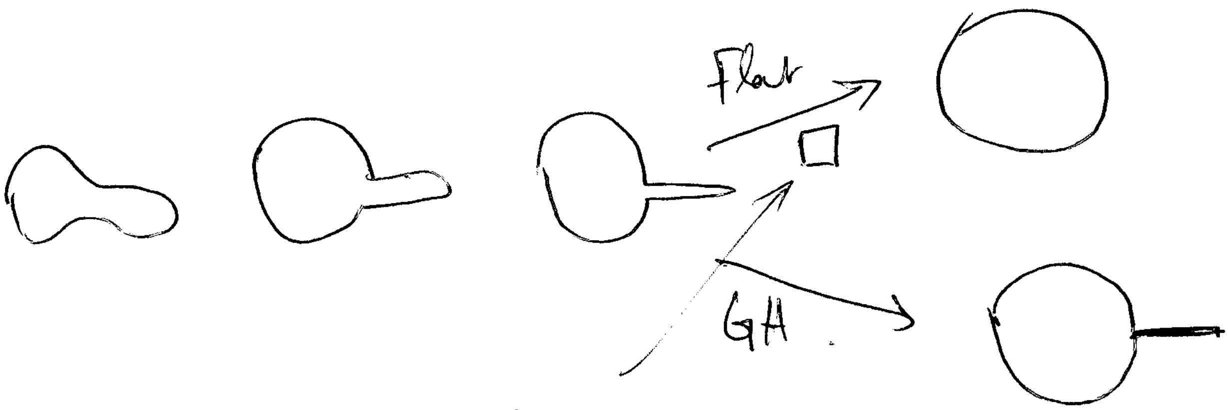
But Flat doesn't see what splines.

• <u>Orientation</u> is really important, because we need to subtract as currents.

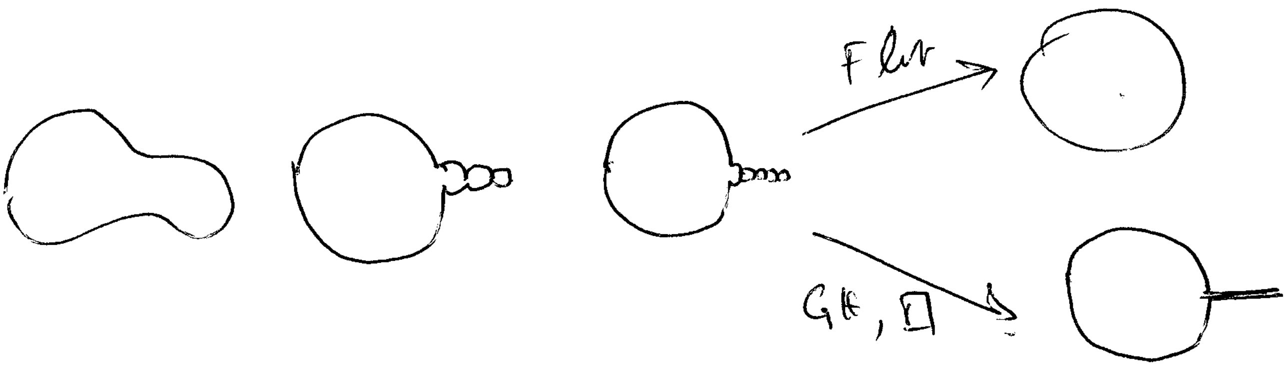
$d_F(M,N) = \inf_Z d_F^Z(\varphi_\#[M], \psi_\#[N])$.

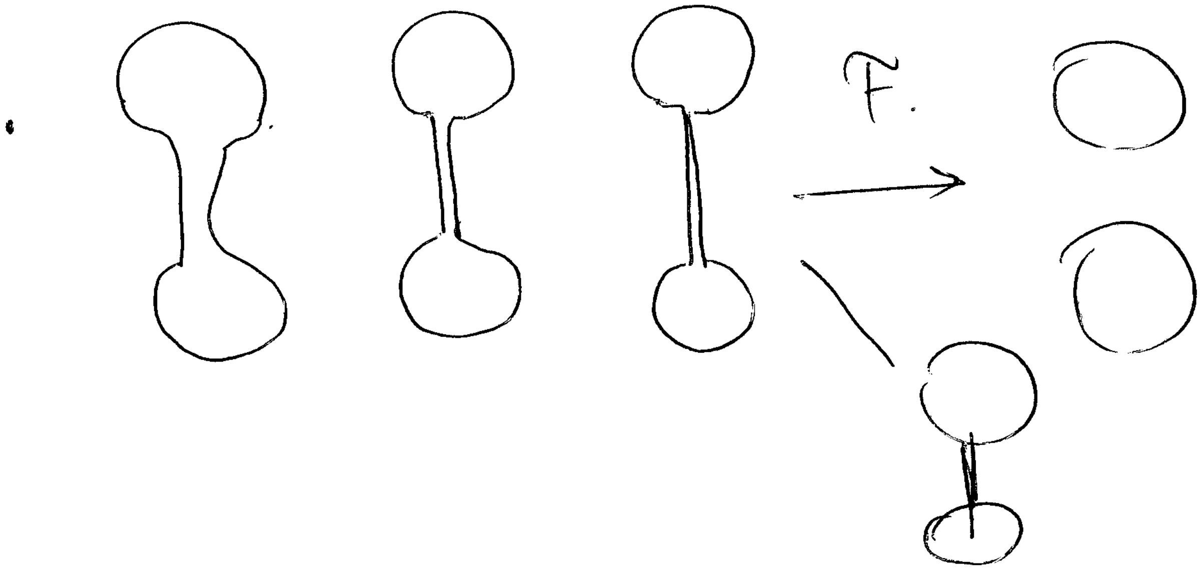
~~$\varphi_\#[M]$~~ $\varphi: M \to Z$. embedding

- **Note:** **Need** a notion of diff form in metric spaces!



Gromov □ distance (using extra measure too).





• In this example, $d_F \not\in d_{GH}$. Ie, invisible line segment, F-limit fails to be geodesic space in general

Integral Current Spaces

X, countably rect. Borel meas maps $\varphi_i : A \subset \mathbb{R}^k \to X$.

$$\mathcal{H}^k\left(X \setminus \bigcup \varphi_i(A_i)\right) = 0.$$

• φ_i can be chosen Bi Lipschitz, with $\varphi_i(A_i) \cap \varphi_j(A_j) = \emptyset$.

• Orientation: preferred charts → bi Lipschitz admit determinant, Ie, pos. determinant.

• Charts don't overlap, so orientation always exists, and the orientation is not quite free. It's non-unique, arbitrary.

• Multiplicity $\theta : X \to \mathbb{R}_+$. Borel.

Defn k-dim. integral current space (X, d, T).

countably $\mathcal{H}^k$-rectifiable (X, d), with current structure $T \in I_k(\bar{X})$, $\text{set}(T) = X$.

Integral current T has same dimension as X.

• $\mathrm{spt}(T) = \{y : \mathrm{dens} > 0\}$

$$\mathrm{dens}(y) = \liminf_{r\to 0} \frac{M(T \llcorner B(y,r))}{r^k} > 0.$$

• Integral current space - is a rectifiable metric.
Singularities $M(T \llcorner B(y,r)) = \mathrm{vol}(B(y,r))$.

Def^n Flat for ICS:

$$d_F(M_1, M_2) = \inf d_Z^F(\varphi_{1\#}T_1, \varphi_{2\#}T_2).$$

$$\underbrace{\varphi_1 : X_1 \to Z,\ \varphi_2 : X_2 \to Z}_{\text{metric only comes in here}}.\quad \text{isometric}.$$

• Rmk: the need for ICS is b/c we want limits to also ~~be~~ admit bi-Lipschitz charts countably etc..

• When no thin bits or cancellation, GH limit and IFC agree. (This is the existence of a uniform linear contractibility fn).

• Cheeger-Colding (Ricci $\geq$ —), the Flat limit agrees! But no uniform linear contractibility here, so doesn't follow from prev. results.

- IF needs isometric embedding. Often, need to go into much higher dimension to get this.