Optimal Transport and metric Sobolev Spaces. 02/03/2015.

Savaré.

Problem: define $W^{1,2}(X)$. w/o doubling/Poincaré.

Upper semi-cts slope: ~~$D^* f(x)$ / pt-wise Lipschitz~~

$$|D^* f|(x) = \lim_{r \downarrow 0} \sup_{y,z \in B_r(x)} \frac{|f(z) - f(y)|}{d(z,y)} \quad \text{for Lipschitz}$$

$$H^{1,2}(X,d,m) = \{ f \in L^2(X,m) : \exists f_n \in Lip(X),\ f_n \to f \in L^2(X,m), \int |D^* f_n|^2 \, dm \leq C < \infty \}.$$

Cheeger energy: $\mathcal{E}(f) = \inf \{ \liminf_{n \to \infty} \int_X |D^* f_n|^2 \, dm : f_n \in Lip(X), f_n \to f \in L^2(M) \}$.

If f realises energy, define $|Df|_*$ as $\lim |Df_n|^2 \, dm$.

Then, can write $\mathcal{E}(f) = \int_X |Df|_*^2 \, dm$ if $f \in H^{1,2}$; $+\infty$ otherwise.

2 hom-convex, l.s.c.

More importantly Newtonian Spaces.

Upper gradient: Borel $G: X \to [0,\infty]$.

$$\left| \int_{\partial \gamma} f \right| \leq \int_\gamma G. \quad \forall \gamma \in \mathcal{C}(X)$$

abs. cts curve.

If G $\int_\gamma G < \infty \Rightarrow$ for γ AC.

<u>Note:</u> UG only depends on length structure of d.

i.e., $d_\ell(x_0,x_1) = \inf\{\ell(\gamma)\} \geq d(x_0,x_1)$.

does not <u>affect</u> UG.

Newtonian Space: $N^{1,2}(X,d,m) := \{ f : X \to \mathbb{R} : \int_X |f|^2 \, dm < \infty,$
$\exists$ UG of f w.r.t. $\int_X G^2 \, dm < \infty$.

$\|f\|_{N^{1,2}} := \inf \{ \int |f|^2 + \int G : \exists G \in UG(f) \}$.

not independent of negligible modifications of f.

<u>Fuglede's lemma:</u> $G_n \to g$ strongly; then $\exists\, n(k)$ subsequence.
$\int_\gamma G_{n(k)} \to \int_\gamma g$ Mod-a.e. γ.

$G_n \in UG(f)$, then g weak upper grad.
$\left| \int_{\partial\gamma} f \right| \leq \int_\gamma g$

* Modulus of a family of curves.

$\text{Mod}(\Pi) = \inf \{ \int_\gamma g^2 : \int_\gamma g \geq 1 \ \forall \gamma \in \Pi \}$

behaves like measure.