Ambrosio. Ext. Met. meas. spaces + Ricci curv. 02/08/2015.

Extended: $d : M \times M \to [0, +\infty]$. Ie, $d(x,y) = +\infty$ is cool.

Why? (I). W_2, in general, is extended.
→ connected components and usual $d < \infty$

(II) Wiener space.

(III). $\mathcal{E}$ Dirichlet form, then

$$d_{\mathcal{E}}(x,y) = \sup \{ |f(x) - f(y)| : \mathcal{D}(\mathcal{E}) \cap C_b(X),$$
$$\Gamma(f) \leq m\text{-a.e. in } X \}.$$

(Lipschitz? What is ?).

(IV) Configuration. spaces → ie path spaces (Wiener).

* Γ - carré du champ.

Strong Feller assumption on. semigroup. P_t, ie

$$P_t : L^\infty(X, m) \to C_b(X).$$

* see slides for Thm. equivalence b/w. contractivity, K-convexity, etc.

Ricci Curvature [AGS.] $\mathrm{Ch}(f) = \frac{1}{2} \int |Df|^2 \, dm$ quadratic

⟹ EVI_K encodes. strong convexity of Ent, Cheeger quadratic and Strong Feller

Extended problems. W. closure may not be extended, and may not include Dirac masses.

Def^{n}. (X,τ,d) extended m.m.s:

(I) $d: X\times X \to [0,\infty]$ extd. distance.

(II) τ topology on X.

(III) $\exists\ d_n$, τ-contin., $d_n \uparrow d$.

No assumption that d convergence $\Rightarrow$ τ convergence.

But notion of tightness to compensate.

Assumption (III) in def^n not restrictive; if (X,τ) Polish, then d_ε automatically satisfies (III).

$\mathrm{Lip}_b(X,\tau,d) = \{f \in C_b : \text{———}\}$.

~~obtain~~ $Ch(f)$ as L^2 relaxation, but then.

Define then. $Ch(f) := \frac{1}{2}\int |Df|_w^2\, dm$, $\forall f\in \mathcal{D}(Ch)$.

(*) Obtain Cheeger energies $Ch_n(f)$ w.r.t $d_n \uparrow d$.

See Th^m on slide about Ch_n as $L^2(X,m)$-l.s.c. envelope of $\inf_n Ch_n$.

$Ch_{n+1} \leq Ch_n$.

Capacity: (Newtonian).

$$\mathrm{Cap}(E) := \left\{ \inf \|u\|^2_{N^{1,2}} : u \geq 1 \text{ on } E \right\},$$

$\mathrm{Cap}(E) = 0 \iff$ ______ ~~modul~~ condition on $\gamma^{-1}(E) \neq \emptyset$.

$N^{1,2}$: m-measurable, $\int_X f^2 < \infty$ and $\exists$ w.u.g. g.

<u>Sobolev Space via dynamic plans:</u>

Consider probability measures π on $AC([0,1], X)$.

a "dynamic plan". Test plan $\pi \in \mathcal{P}(AC)$.

$$E = \int \mathrm{Energy}(\gamma)\, d\pi(\gamma) = \iint_0^1 |\dot\gamma_t|^2\, dt\, d\pi(\gamma) < \infty.$$

and $\exists\, C > 0$ s.t.

$(e_t)_\# \pi \leq C m$, $\quad \pi[\gamma : \gamma_t \in A] \leq C m(A)$

$e_t(\gamma) = \gamma_t$.

~~Barycenter~~

<u>π has Barycenter in L^2.</u>

$\mu \in AC^2([0,1]; X)$ ~~then~~ iff $\exists$ dynamic plan s.t.

$$(e_t)_\# \pi = \mu_t \;; \quad \iint_0^1 |\dot\gamma_t|^2\, dt\, d\pi(\gamma) < \infty.$$

and there is an optimal one. to min

$\int_0^1 |\dot\gamma_t|^2\, dt$.

$\Gamma \subset AC^2$ negligible if $\pi(\Gamma) = 0$ $\forall$ test plan π.

$f = h$ m-a.e. $\Rightarrow \int_\gamma f = \int_\gamma h$.

weak gradient

$$\left| \int_{\partial\gamma} f \right| \leq \int_\gamma g \quad \text{a.e. curve } \gamma.$$

weak gradient exists $\Rightarrow$ ~~weak~~

$$W^{1,2}(M) = \{ f : \int_\gamma (f^2 + |\nabla f|^2) < \infty \text{ a.e. } \gamma \}$$

$\|\cdot f\|_{W^{1,2}} \quad \int |f|^2 + |\nabla f|^2$.

Thus weak $W^{1,2}(M) \supset N^{1,2}(M)$.

By this theorem, choose "good" ~~representation~~ Schwarz representation. $\Rightarrow N^{1,2}(M) \subset W^{1,2}$.

Wasserstein picture

$$\mathrm{Ent}(\mu) = \int_X n \log n \, dm \quad \mu = nm \in P(X).$$

$|D^- \mathrm{Ent}| \leftarrow$ Fisher information.

Fisher information. $\mathrm{Fish}_X(\mu) := \inf \{ \liminf_{n\to\infty} |D^- \mathrm{Ent}|^2(\mu_n).$

$$\mu_n \to \mu, \ \mathrm{Ent}(\mu_n) \leq C \}$$

$$\mathrm{Fish}_X(\mu) = 4 \mathcal{E}(\sqrt{n})$$
$$= 4 \int_X |D\sqrt{n}|_X^2 \, dm$$
$$= \int_{n>0} \frac{|Dn|_X^2}{n} \, dm.$$

Comparison ~~to~~ classic via Gradient flows.