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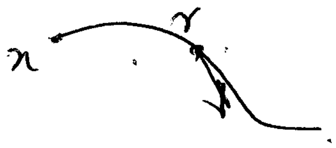
Maximal flows of non-smooth v-fields + apps.

Flow: $b:(0,T)\times\mathbb{R}^d \to \mathbb{R}^d$.



$\dot\gamma = b(\gamma)$, i.e.:

$$\begin{cases} \partial_t X(t,x) = b_t(X(t,x)) \\ X(0,x) = x. \end{cases}$$

Continuity eq^n: $\partial_t \mu_t + \operatorname{div}(b_t \mu_t) = 0$

But we take μ_t to be finite measures in $\mathbb{R}^d$ via distributional interpretation.

Transport eq^n:

$$\begin{cases} \operatorname{div} b_t = 0 \\ \partial_t \mu_t + b_t \cdot \nabla \mu_t = 0. \end{cases}$$

Connection b/w flows and CE "sol^n of CE are transported by the flow": $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$.

$$\mu_t := X(t,\cdot)_\# \mu_0.$$

Solves the CE.

φ test func.

$$\frac{d}{dt}\int \varphi \, d\mu_t = \frac{d}{dt}\int \varphi(x)\, d\mu_0 = \int \nabla\varphi(x)\cdot h_t(x)\, d\mu_0 = \int \nabla\varphi \cdot h_t \, d\mu_t .$$

<u>Physical motivation:</u> The Vlasov-Poisson eq^n.

Consider the N-body problem:

$$\begin{cases} \ddot{x} = F_t(x). \\ F_t = \nabla u_t , \quad u_t \text{ - gravitational potential.} \end{cases}$$

$$\Leftrightarrow \quad \begin{matrix} \dot{x} = v . \\ \dot{v} = F_t(x) . \end{matrix} \Leftrightarrow \quad h_t(x,v) = (v, F_t(x)).$$

(Moving under its own force, ie galaxy, electrons in plasma etc.)

Makes sense to describe this in terms of density distribution with velocity v and pos. x — $f_t(x,v)$.

~~$f_t(x,v)$~~

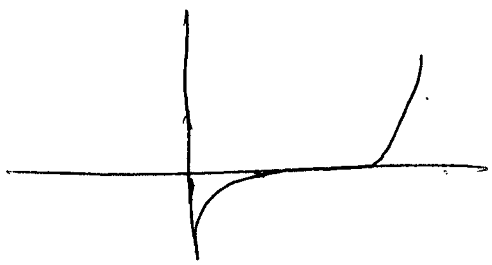
f_t solves CE. $\quad \partial_t f_t + \operatorname{div}(h_t f_t) = 0.$

$$\begin{cases} \partial_t f_t + v \nabla_x f_t + F_t(x) \nabla_v f_t = 0. \\ \rho_t = \int f_t \, dv . \\ F_t := \int \frac{x-y}{|x-y|^d} \rho_t(y)\, dy . \end{cases}$$

$\Leftrightarrow \quad \Delta u_t = \rho_t , \quad F_t = \nabla u_t .$

Classical theory. b_t loc. Lip. $\Rightarrow$ $\exists!\ X(\cdot,x) : [0, T_x(x)) \to \mathbb{R}^d$ sol^{n}. of ODE.

Nonsmooth theory. $\leftarrow$ miss uniqueness; Ex. $b(x) = \sqrt{|x|}$, $x \in \mathbb{R}$.



Def^{n}. X is a Lagrangian flow for b if. (RLF).

① $X(\cdot, x)$ is AC and solves the ODE.

② $(X(t,\cdot))_{\#} \mathcal{L}^d \le C \mathcal{L}^d$ (does not concentrate Lebesgue measure).

Th^{m}. [Di P-Lions '89, Ambrosio '04].

$b_t \in W^{1,1}_{loc} / BV_{loc}$, $|\mathrm{div}\, b_t| \le C$, $\frac{|b_t(x)|}{1+|x|} \in L^\infty + L^1$;

then. $\exists!$ RLF for b.

Difference from Classical: classical is local - ie, blow up of trajectory.

Local non-smooth theory

Pf of prev Th$^{\underline{m}}$:

1. Existence by approximation.
2. Well posedness of PDE $\Rightarrow$ uniqueness of flow. (in the class of bdd, non-neg, cpct spt. sol$^{\underline{n}}$s.)
3. Prove the well posedness.

Improved Th$^{\underline{m}}$:
- b locally integrable.
- $\operatorname{div} b_t \in L^\infty_{loc}$.
- $b_t \in W^{1,1}_{loc}$.

$\exists!\ X(\cdot, x) : [0, T_x(x)] \to \mathbb{R}^d$ Maximal Flow of b,

- Maximal: — cannot extend trajectory, ie

  $\limsup |X(t,x)| = \infty$ as $t \to T_x(x)$.

- Regular: $X(t,\cdot)_\# \mathcal{L}^d \llcorner \{T_x > t\} \lesssim \mathcal{L}^d$.

- Flow: $X(\cdot, x) \in AC_{loc}([0, T(x)])$ and solves the ODE.

Back to Vlasov-Poisson system

Problem $f_t \in L^1 \not\Rightarrow E_t f_t \in L^1_{loc}$

But u solves TE ($\Leftrightarrow$ CE) $\Rightarrow \beta(u)$ solves TE.

$$\partial_t \beta(u) + b_t \nabla \beta(u) = \beta'(u) \underbrace{[\partial_t u + b_t \nabla u]}_{\overset{\text{"}}{0}} = 0$$

Def^n $f_t \in L^1$ is a renormalized sol^n of VP if

(VP1) is satisfied by $\beta(f_t)$. $\forall \beta \in C^1 \cap L^\infty$.

Thm $f_t \in L^1 \cap L^\infty$ distrib sol^n of VP. Then f_t is

transported by the measured regular flow of $b_t(x,v) = (v, F_t(x))$.

$\Updownarrow$

$f_t(\gamma(t)) = \text{const}$. $\forall \gamma$ in the Max. Reg. Flow. (MRF).

Entropy $(f_t) = \int f_t \log f_t = \text{const}$.

Theorem. Existence of weak sol^{n}s under very gen.

$f_0 \in L^1$, $\int |v|^2 f_0 \, dx \, dv + \int \rho_0 \, g(x) \, dx < \infty$.

Then $\exists$ a renormal sol^n of VP.