David Rule — Bilinear Osc. 28/07/2015.

Bilinear oscillatory integral.

$$T_{a,\varphi}(f,g)(x) = \iint a(x,\xi,\eta)\,\hat f(\xi)\,\hat g(\eta)\,e^{i\varphi(x,\xi,\eta)}\,d\xi\,d\eta.$$

Motivation: [Bernicot–Germain '10]:

$$\partial_t f + iP(D)f = T_a(f,f), \quad P(D) = -\Delta.$$

i.e. nonlinear Schrödinger,

+ Gravitational water waves, products of sol^ns to Euclidean waves.

Th^m. (a la Seeger–Sogge–Stein) [Rodríguez-López, R., Staubach]:

- $\varphi(x,\xi,\eta) = \varphi_1(x,\xi) + \varphi_2(x,\eta)$.
- $|\partial_\xi^\alpha \partial_\eta^\beta \partial_x^\gamma a(x,\xi,\eta)| \le C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m-(|\alpha|+|\beta|)}$.
- $\operatorname{spt}_x a(x,\xi,\eta)$ cpct., $\varphi_j$ hom. of deg 1.

  $\det \nabla_x \nabla_\xi \varphi_j(x,\xi) \neq 0$ on spt $a$.

Get $L^p$ boundedness.

Pf. By application of 1-dim results. But cutoff:

$$T(f,g) = \underbrace{T(\chi(D)f,g) + T((1-\chi)(D)f,\chi(D)g)}_{\text{low-freq.}} + \underbrace{T((1-\chi)(D)f,(1-\chi)(D)g)}_{\text{high freq.}}.$$

- Low freq: Iteration yields $L^p \times L^q \to L^r$, $1 \le p,q \le \infty$.
- High freq: Coifman–Meyer's techniques for pseudos in $OPS^m_{1,0}$ ← need $m < 0$ for endpoint estimates.