Stochastic methods in Rayleigh–Bénard Convection models : Geordie Richards. 27/07/2015.

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla p - \nu_1 \Delta u = g\theta \hat{x}_d \; ; \quad \nabla\cdot u = 0 . \\ d\theta + (u\cdot\nabla\theta - \nu_2\Delta\theta)\,dt = \sigma\, dW \\ (u,\theta)\big|_{t=0} = (u_0,\theta_0) . \end{cases}$$

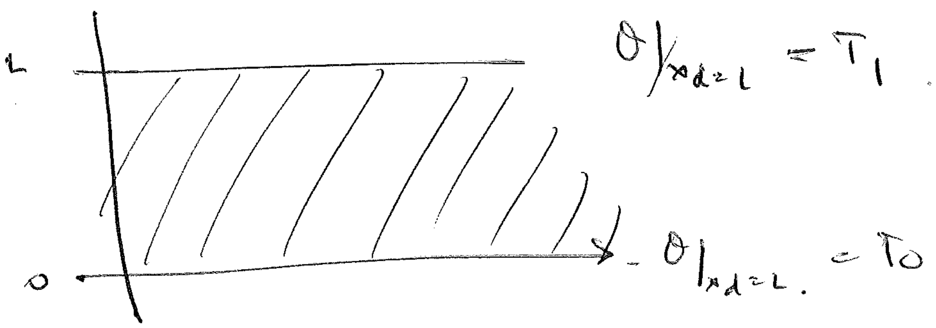
unit vector in vertical direction.

$0 \leq x_d \leq L$

$(x_1, \dots, x_{d-1}) \in \mathbb{T}^{d-1}$. periodic.

$x = (x_i) \in \mathbb{R}^d \quad d = 2, \text{ or } 3. \quad t \geq 0, \; \nu_1, \nu_2 > 0 .$

$u(t,x) := (u_i(t,x)) \in \mathbb{R}^d, \; \theta(t,x)$ temperature of fluid. ~~fluid~~.



$T_0 \gg T_1 \Rightarrow$ turbulent convection. poorly understood when $Pr = \frac{\nu_1}{\nu_2} \gg 1$.

Rescaling,
$$\begin{cases} \frac{1}{Pr}\left(u_t + (u\cdot\nabla)u\right) + \nabla p - \Delta u = Ra\,\theta \hat{x}_d , \quad \nabla\cdot u = 0 \\ d\theta + (u\cdot\nabla\theta - \Delta\theta)\,dt = \sigma\, dW . \end{cases}$$

Pr – "Prandtl number?", Ra – Rayleigh const.

$0 \leq Ra , \quad 0 \leq x_d \leq 1 .$

These equations are physical, happen in the sun etc.
Main eq without $g\theta\hat{x}_d$ is Navier–Stokes.

Stat. prop. of turbulent flow should converge to an equilibrium as $t \to \infty$, indep of initial cond.

* Want to know whether there is an ergodic measure.

Th^m 1. [Földes, Glatt-Holtz, R.; Thomann, '13]

$x \in (x_1, x_2) \in \mathbb{T}^2 \Rightarrow \sigma\, dw = \sum_{k \in \underline{K}} \alpha_k e^{ik\cdot x} dB_k$, $\alpha_k \neq 0$,

$\{B_k(t)\}$ = seq. of ind Brownian motions,

$k \in \mathbb{Z}^2$ a (possibly) finite set. Assume that

$\{(1,0), (0,1)\} \subset \underline{K}$ then the system

posses a unique invariant ergodic measure μ.

Pf. Based on adapting Hörmander's theory of hypoellipticity to infinite dimensions (see [Hairer-Mattingly '06, '08] for stochastic Navier-Stokes).

~~For $\varphi: H \to \mathbb{R}$.~~

In general, given a stochastic PDE.

$$(\ast) \quad \begin{cases} du + F(u)\, dt = \sigma\, dw. \\ u(t=0) = u_0 \in H. \end{cases}$$

For $\varphi: H \to \mathbb{R}$, letting $P_t \varphi(u_0) = \mathbb{E}(\varphi(u(t, u_0)))$, to prove "unique ergodicity", prove smoothing properties.

for PDE. $V(t, u_0) = P_t \varphi(u_0)$.

$$\begin{cases} \partial_t V = \frac{1}{2} \mathrm{Tr}\left[(\sigma\sigma^*) D^2 V\right] - \langle F(u), DV \rangle. \\ V(t=0) = \varphi. \end{cases}$$

K finite $\Rightarrow$ degenerate dissipation.

Smoothing property means the estimate:

$$\| D P_t \varphi \| \leq C \|\varphi\| + \delta(t) \|D\varphi\|, \text{ where } \delta(t) \to 0 \text{ as } t \to \infty.$$

For $F, G: H \to H$, $[F, G] = DF(G) - DG(F): H \to H$.

$e_k = e_k(x) = e^{ik\cdot x}$.

Navier Stokes: $[[F(u), e_k], e_j] \sim e_{k+j} + e_{k-j}$.

N.S. nonlinearity.

For Boussinesq: $[[F(u,\theta),(0,e_k)],(0,e_j)] = 0$.

But: $[[[F(u,\theta),(0,e_k)], F(u,\theta)], (0,e_j)] \sim (0, e_{k+j} + e_{k-j})$

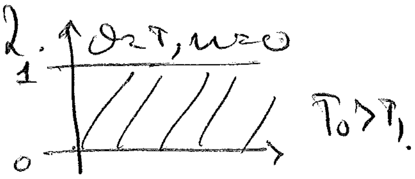
and $[F(u,\theta),(0,e_k)] = (e_k, J_k(u,\theta))$. use to remove the "junk" low frequencies.

Certainly dependent on structure in Boussinesq.

$\boxed{\text{Note: } x\in\mathbb{T}^2 \text{ is not the correct bdy condition}}$

Rmk Hairer–Mattingly generalised Hörmander to infinite dim, but their hypothesis for brackets could not be satisfied in this situ. This is a further weakening of that, i.e. Hörmander cond. has weakened to a lesser class of brackets for which Boussinesq can be fed ~~so as~~ to satisfy hypotheses.

Thm 2. [Földes, Glatt-Holtz, R., '15]. Let $d=2$.



$\sigma dW = \sum_{k=1}^{N} \alpha_k \sigma_k dB_k$, $\alpha_k \neq 0$.
basis for for H.

If Pr and $N = N(Ra, Pr, T_0 - T_1)$ are both sufficiently large, the system posses a unique ergodic invariant measure μ_{Pr}

Pf via some classical techniques, not Hypoelliptic forcing as in Hörmander. Force large number of modes. ("essentially elliptic")

When $Pr \to \infty$, MPE becomes

$$\begin{cases} \nabla p - \Delta u = Ra \cdot \theta \hat{x}_d \\ d\theta + (u \cdot \nabla \theta - \Delta \theta)\, dt = \sigma\, dW \\ \theta(t=0) = \theta_0 \end{cases}$$

Thm 3 [Földes, Glatt-Holtz, R., '15]. ~~Pf~~

If $N = N(Ra, T_0 - T_1)$ is suff large, then the system has a unique ergodic invariant measure μ_∞.

Moreover, letting $\{\mu_{Pr}\}_{Pr \geq 1}$ sequence of invariant measures for Thm 2, $\exists\, q \in (0,1)$ s.t. $\forall$ suff smooth $\varphi: H \to \mathbb{R}$.

$$\left| \int \varphi(\theta)\, d(\Pi_\theta \mu_{Pr}) - \int_H \varphi(\theta)\, d\mu_\infty \right| \leq C_\varphi (Pr)^{-q}.$$