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Paracontrolled Calculus.

$$\partial_t u + Lu = u \cdot \xi \leftarrow \text{white.}$$

~~$\xi : \mathbb{R}^n \to \mathbb{R}$~~

So, $\xi \in C^{-d/2}$ Q: How to define product? Sol^n expected to belong to $C^{-d/2+2}$.

Assumption. $(e^{-tL})_t$ has h.k. with phase upper Gaussian estimates, as well as a "Carré du champ" $(\sqrt{t}\,\Gamma e^{-tL})$ $\Gamma f - |\nabla f|$.

Time-freq. fn. operators $P_t \sim e^{-tL}$, $Q_t \sim (tL)^N e^{-tL}$.

Product as paraproducts:

$$f = \int_0^1 Q_t f \,\frac{dt}{t} + P_1(f).$$

$$\Pi_g(f) = \int_0^1 Q_t[(Q_t f)(P_t g)] \,\frac{dt}{t}. \quad \text{"Paraproduct Part".}$$

$$\Pi(f,g) = \int_0^1 P_t\, Q_t f\, Q_t g \,\frac{dt}{t}. \quad \leftarrow \text{due to "Carré du champ" structure. "Resonant part".}$$

Decomposition:

$$fg = \Pi_g(f) + \Pi_f(g) + \Pi(f,g) + \Delta_1(f,g).$$

I.e., consider $\underbrace{\int_0^1 Q_t[P_t f\, P_t g]}\,\frac{dt}{t}$.

$$Q_t[P_t f\, P_t g] = (tL)^N P_t [P_t f\, P_t g]. \quad \text{By Carré du Champ!}$$

$$\Rightarrow L(P_t f\, P_t g) = \Gamma(P_t f)\, P_t g + \cdots$$

Problem $\quad \|\Pi(f,g)\|_{C^{\alpha+\beta}} \lesssim \|f\|_{C^\alpha} + \|g\|_{C^\beta}$.

if $\alpha+\beta>0$. Cannot define $f\cdot g$ for $d\geq 2$.

Notion of Paracontrolled: $\quad X \in C^\alpha$ "well-known",

$f\in C^\alpha$ "paracontrolled" if $\exists\, g\in C^\alpha$ s.t.

$$f = \pi_g(X) + f^\#.$$

where $f^\# \in C^{2\alpha}$.

Three commutator estimates: $\quad f\in C^\alpha,\ g\in C^\alpha,\ h\in C^\gamma$.

$$\pi(\pi_f(g),h) - f\cdot\Pi(g,h) \in C^{2\alpha+\gamma}.$$

if $\alpha+\alpha+\gamma>0$ (improvement from $\alpha+\gamma>0$).

Picture: $\left[\begin{array}{l} (\partial_t+L)u = u\cdot\xi, \quad \text{paracontrolled by } X \\ \underbrace{(\partial_t+L)X = \cancel{\xi}\ \xi}_{\text{only given by white noise, so known!}} \end{array}\right.$

Need $\Pi(X,\xi)$ to be cts. $/ C_T C^{2\alpha-2}$ to make this work. Need to "renormalize" white noise.

But $\Pi(X,\xi)$ defined a.e.

Let $\xi^\varepsilon = e^{-\varepsilon L}\xi$, and via approximation procedure,

find c^ε s.t. $\Pi(X^\varepsilon,\xi^\varepsilon) - c^\varepsilon \to 0$ in $C_T C^{2\alpha-2}$.

Use this to construct approx. sln's and have them

converge in $C_T C^{2\alpha-2}$.