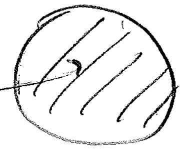
The cubic Szegő equation: Integrability and Turbulent sol^{ns}. P. Gérard 29/07/2015.

$L^2_+(S^1) = \{ u(e^{ix}) = \sum_{k\geq 0} \hat{u}(k)\, e^{ikx},\ \sum_k |\hat{u}(k)|^2 < \infty \} \subset L^2(S^1)$

$u(z) = \sum_{k\geq 0} \hat{u}(k) z^k \in \text{Hardy}^2(\mathbb{D})$.



$\Pi : L^2 \to L^2_+$. orthogonal projector.

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u). \\ u|_{t=0} = u_0. \end{cases}$$

globally well posed on $H^s_+(S^1) := H^s(S^1) \cap L^2_+(S^1)$, $s \geq \frac{1}{2}$.

$\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, $\|u(t)\|_{L^4} = \|u_0\|_{L^4}$, $(Du(t), u(t))_{L^2} = (Du_0, u_0)_{L^2}$.

(I). Hankel operators & Lax Pair.

$\ell^2(\mathbb{N}) = \{ \underline{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}},\ \sum_{n=0}^{\infty} |x_n|^2 < \infty \}$.

$\underline{c} = (c_n) \in \ell^2$ $\quad \Gamma_c : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$. $(x_n) \mapsto (y_n)$.

$$y_n = \sum_{p=0}^{\infty} c_{n+p}\, x_p .$$

$S : (x_0, x_1, \dots) \longmapsto (0, x_0, x_1, \dots)$, $S^* = (x_0, x_1, \dots) \mapsto (x_1, x_2, \dots)$

$$S^*\Gamma_c = \Gamma_c S = \Gamma_{S^*c} .$$

Typically unbounded, boundedness Z. Nehari (1957).
Γ_c bdd iff $\exists f \in L^\infty(S^1)$ $\forall n \geq 0$, $c_n = \hat{f}(n)$.

$$\operatorname{Tr}(\Gamma_c\Gamma_c^*) = \sum_{n,p} |c_{n+p}|^2 = \sum_{k=0}^{\infty} (k+1)|c_k|^2 .$$

$$\sum_{p=0}^{\infty} |c_p| \leq \operatorname{Tr}(|\Gamma_c|) = \operatorname{Tr}\left(\sqrt{\Gamma_c\Gamma_c^*}\right) \leq \sum_{p=0}^{\infty}\left(\sum_{n=0}^{\infty} |c_{n+p}|^2\right)^{\frac{1}{2}}.$$

Realisation on $L^2_+(\mathbb{T})$: $u \in L^2_+(\mathbb{T})$, $H_u : L^2_+ \to L^2_+$, $H_u(h) = \Pi(u\bar{h})$.

$$\widehat{H_u(h)} = \Gamma_{\hat{u}}(\bar{\hat{h}}), \quad H_u^2 \leftrightarrow \Gamma_u \Gamma_u^*,$$

H_u Hilbert-Schmidt iff $u \in H^{\frac{1}{2}}_+$

$$\|u\|_{L^\infty} \le \mathrm{Tr}|H_u| \lesssim C_s \|u\|_{H^s}, \quad s > 1.$$

Toeplitz operators $b \in L^\infty(S^1)$, $T_b : L^2_+ \to L^2_+$, $h \mapsto \Pi(bh)$.

Th^m (S. Grellier, PG, 2010).

If $u \in C(\mathbb{R}, H^s_+)$, $s > \frac{1}{2}$ is a sol^n to $i\partial_t u = \Pi(|u|^2 u)$.

Then, $\dfrac{dH_u}{dt} = [B_u, H_u]$, $B_u = -i T_{|u|^2} + \frac{i}{2} H_u^2$.

Pf. Lemma: $a, b, c \in L^\infty_+ = L^\infty \cap L^2_+$, $H_{\Pi(a\bar{b}c)} = H_a T_{b\bar{c}} + T_{a\bar{b}} H_c - H_a H_b H_c$.

$$\partial_t u = -i\Pi(|u|^2 u), \quad \partial_t H_u = -i H_{\Pi(u\bar{u}u)}$$

$$= i(H_u T_{|u|^2} + T_{|u|^2} H_u - H_u^3).$$

$$= H_u(i T_{|u|^2}) - i T_{|u|^2} H_u + \frac{i}{2} H_u^2 H_u - H_u\left(\frac{i}{2} H_u^2\right).$$

$$= \left[-i T_{|u|^2} + \frac{i}{2} H_u^2, H_u\right] = [B_u, H_u].$$

Corollary. $\exists\, U = U(t)$ unitary operators on L^2_+,

$$H_{u(t)} = U(t) H_{u(0)} U(t)^*.$$

Pf. Solve linear ODE, $\partial_t U = B_{u(t)} U(t)$. $U(0) = I$.

and compute: $\partial_t [U(t)^* H_{u(t)} U(t)] = 0$, $\partial_t (U(t)^* U(t)) = 0$

using: $B_u^* = -B_u$.

Corollary 2. If $u_0 \in H^s_+$, $s>1$, then the sol^n $u(t)=Z(t)u_0$ of $i\partial_t u = \Pi(|u|^2u)$ with $u(0)=u_0$ satisfies.

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty} \le C_s \|u_0\|_{H^s}.$$

$$\|u(t)\|_{L^\infty} \le \mathrm{Tr}\,|H_{u(t)}| = \mathrm{Tr}\,|H_{u_0}| \le C_s \|u_0\|_{H^s}.$$

$$u(t) = u_0 - i\int_0^t \Pi\left(|u|^2u(\tau)\right)d\tau, \quad \cancel{\|u(t)\|_{H^s} \le \|u_0\|_{H^s} e^{C_s}}$$

$$\|u(t)\|_{H^s} \le \|u_0\|_{H^s}\, e^{C_s^2 \|u_0\|_{H^s}^2 |t|}.$$

The second Lax pair and finite rank sol^{n}s.

$S = T_{e^{ix}}$, $S^* = T_{e^{-ix}}$, $S^*H_u = H_uS = H_{S^*u} =: K_u$.

Th^m. If u is a sol^n to Szegő, $\partial_t K_u = [C_u, K_u]$,

$$C_u = -iT_{|u|^2} + \frac{1}{2}K_u^2$$

apply S^* on left of the previous Lax pair formula.

$$K_u^2 = H_u^2 - (\cdot|u)u.$$

$$K_u^2 = H_u SS^* H_u \qquad \left[\begin{array}{l} SS^* = I - (\cdot|1)1. \\ H_u(1) = u. \end{array}\right.$$

$$\mathrm{Tr}(H_u^2) = \sum(\ell+1)|\hat{u}(\ell)|^2 = \|u\|_{L^2} + (|D|u|u).$$

Eigenvalues of H_u^2, K_u^2:

(s_j^2) $\quad$ $(s_k')^2$. $\quad s_1 \ge s_1' \ge s_2 \ge s_2'$.

conservation laws!

Consider for every $d \in \mathbb{N}$,

$$\mathcal{V}(d) = \{u : \mathrm{rk}(H_u) + \mathrm{rk}(K_u) = d\}$$

Kronecker (88) $= \{u(z) = \dfrac{A(z)}{B(z)}$; A, B poly. of common factor, $B \ne 0$ on $\mathbb{D}$; $B(0)=1$, and

and $\simeq \Omega\mathbb{C}^{d}$ if $\left.\begin{array}{l} d=2N, \ \deg A \le N-1, \deg B = N. \\ d = 2N-1 \ \deg A = N-1, \deg B \le N \end{array}\right\}$

complex manifold, equation ODE w.r.t. to Kähler structure!

Def. $\mathcal{U}_{gen}(d) = \{ u \in \mathcal{U}(d) : s_1 > s_1' > s_2 > \dots \}$.

Lemma. $\mathcal{U}_{gen}(d)$ is open dense in $\mathcal{U}(d)$.

Thm. The mapping $u \in \mathcal{U}_{gen}(d) \xrightarrow{\Phi} ((s_1, s_1', \dots); (\varphi_1, \varphi_1', \dots))$ is a diffeomorphism

"angles" defined on an open.

$\Phi(u) = ((s_1, s_1', \dots), (\varphi_1, \varphi_1', \dots))$.

$\Phi(Z(t)u) = ((s_1, s_1', \dots); (\varphi_1 + t s_1^2, \ \varphi_1' + t(s_1')^2, \dots))$.

"action angle variables".

Furthermore, Φ^{-1} is given by:

$$\mathcal{C}(z) = \left(\frac{s_j e^{i\varphi_j} - z s_k' e^{i\varphi_k'}}{s_j^2 - (s_k')^2} \right)_{1 \le j,k \le N}$$ is invertible. $\forall z \in \mathbb{D}$.

$N \times N$ matrix. $N = \left[\frac{d+1}{2}\right]$. $= (\text{rk } Th^{\underline{m}})$.

$$u(z) = \left\langle \mathcal{C}(z)^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle.$$