Long time Sobolev estimates for sol^{ns} of Ham. PDE — Patrick Gérard.

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(II) Fractional NLS and $\frac{1}{2}$ wave eq^n.

$$(*)_\alpha \begin{cases} i\partial_t u - |D|^\alpha u = |u|^2 u, \quad t\in\mathbb{R},\ x\in\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad 1<\alpha\le 2. \\ E_\alpha(u) = \frac{1}{2}\underbrace{(|D|^\alpha u, u)_{L^2}}_{=\|u\|^2_{\dot H^{\alpha/2}}} + \frac{1}{4}\|u\|^4_{L^4}. \end{cases}$$

a) Global wellposedness + crude Sobolev est. (Recall).

Th^m. $\forall \alpha\in[1,2]$, $\forall s>\alpha/2$, $\forall u_0\in H^s(\mathbb{T})$, $\exists!\, u\in C(\mathbb{R}, H^s(\mathbb{T}))$ sol^n to $(*)_\alpha$ with $u(0)=u_0$.

+ Sobolev est.: If $\alpha>1$, $\|u(t)\|_{H^s} \le B(u_0,s,\alpha)\, e^{B|t|}$.
If $\alpha=1$, $\|u(t)\|_{H^s} \le e^{Be^{B|t|}}$

Because: $H^{\alpha/2}\subset L^\infty$ if $\alpha>1$. $\|u\|_{L^\infty}\le C\|u\|_{H^{\alpha/2}}$.
$H^\alpha \not\subset L^\infty$ if $\alpha=1$, $\|u\|_{L^\infty} \le C_s \|u\|_{H^{\frac12}}\left[\log\left(2+\frac{\|u\|_{H^s}}{\|u\|_{H^{\frac12}}}\right)\right]^{\frac12}$.

These are not optimal.

b) Improved Sobolev estimates:

Theorem (Joseph Thirouin, 2015).

- If $\alpha>1$, $s=\alpha+n$, $n\in\mathbb{N}$, $\|u(t)\|_{H^s}\lesssim C_{n,\alpha}(u_0)(1+|t|)^{N(u_0,\alpha,n)}$

If $\alpha=1$, $n\ge 1$ $\|u(t)\|_{H^n}\lesssim C_n(u_0)\, e^{C_n(u_0)t^2}$.

(Everything depends on initial data u_0).

Rmk. If $\frac{8}{9}<\alpha<1$, global wellposedness + polynomial Sobolev estimates.

Trick: $\| e^{it|D|^\alpha} u_0 \|_{L^4([0,\delta]; L^\infty(\mathbb{T}))} \lesssim C \|u_0\|_{H^{\frac{1}{2}-\frac{\alpha}{4}+\varepsilon}}$.

($\frac{2}{3}<\alpha$ because you want $\longrightarrow$ to be controlled by Energy norm).

c) Further analysis of the case $\alpha = 1$. (Half wave eq^n).

$$\underbrace{(i\partial_t - |D|)}_{\frac{1}{2}\text{-wave op.}}(-i\partial_t - |D|) = \partial_t^2 - \partial_x^2 .$$

Duhamel formulation: $u(t) = e^{it|D|}u_0 - i\int_0^t e^{-i(t-\tau)|D|}\left[|u(\tau)|^2 u(\tau)\right]d\tau$.

$$v(t) := e^{it|D|} u(t),$$

$$\hat{v}(t,k) = \hat{u}_0(k) - \int_0^t \sum_{k_1-k_2+k_3=k} e^{i\tau\omega(k_1,k_2,k_3,k)}\hat{v}(\tau,k_1)\overline{\hat{v}(\tau,k_2)}\hat{v}(\tau,k_3)\,d\tau$$

and $\omega(k_1,k_2,k_3,k) = |k_1| - |k_2| + |k_3| - |k| \in \mathbb{Z}$.

(For α, $\omega_\alpha(k_1,k_2,k_3,k) = |k_1|^\alpha - |k_2|^\alpha + |k_3|^\alpha - |k|^\alpha$).

Assuming $|\hat{v}(t,k)| \sim \varepsilon$, small solutions.

$\omega \neq 0 \Rightarrow$ better terms at infinity by integration by parts. ($\varepsilon \rightsquigarrow \varepsilon^4$).

$\omega = 0 \Rightarrow$ non-oscillation or "totally resonant terms".

Totally resonant system: ~~$\hat{v}(t,k) = \hat{u}_0(k) - i$~~

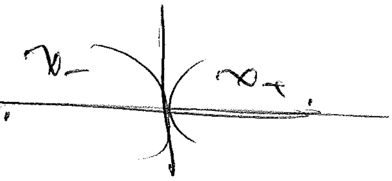
$$\hat{v}(t,k) = \hat{u}_0(k) - i\int_0^t \sum_{\substack{k_1-k_2+k_3=k \\ |k_1|-|k_2|+|k_3|=k}} \hat{v}(\tau,k_1)\overline{\hat{v}(t,k_2)}\,\hat{v}(\tau,k_3)\,d\tau .$$

Lemma. If $k_1 - k_2 + k_3 - k = 0$ and $|k_1| - |k_2| + |k_3| - |k| = 0$.
either (I) $(k_1, k_3) = (k_2, k)$ or (II) k_1, k_2, k_3, k have same sign.

Pf. High-school maths.

(I) → "Trivial resonances". $v(t) = e^{2it\|v_0\|^2_{L^2}} \tilde{v}(t)$.
kills all trivial resonances.

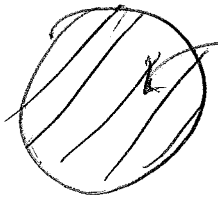
(II). $v_\pm = \Pi_\pm(v)$, $\Pi_\pm = \chi_{\pm D} \geq 0$.



Because you get two regimes depending on sign.

Cubic Szegő eq^{n}s.
$$\begin{cases} i\partial_t v_+ = \Pi_+(|v_+|^2 v_+). \\ i\partial_t v_- = \Pi_-(|v_-|^2 v_-). \end{cases}$$
| $\Pi_\pm$ Szegő or Reisz projector.

Cubic Szegő eq^n $\Pi_+ = \Pi$, $i\partial_t v = \Pi(|v|^2 v)$.



Hardy space $\hat{v}(k) = 0$, $\forall k < 0$.
(this is where sol^n lives).

Th^m. (S. Grellier, PG, O. Pocovnicu).
$u_0 \in H^s$, $\Pi(u_0) = u_0$, $v(t) = e^{it|D|} u(t)$.
$(s > 1)$ $\|u_0\|_{H^s} \approx \varepsilon$, $i\partial_t u - |D| u = |u|^2 u$.

$$i\partial_t w = \Pi(|w|^2 w), \quad w(0) = u_0.$$

$$\|v(t) - w(t)\| \lesssim \varepsilon^2, \quad |t| \leq \frac{C}{\varepsilon^2} \log\left(\frac{1}{\varepsilon}\right).$$

(Rmk this th^m shows it is an approx of sol^n to $\frac{1}{2}$-wave is not too much of a cheat)

(III). The cubic Szegő Eq^{n}.

$$i\partial_t u = \Pi(|u|^2 u), \quad L^2_+(\mathbb{T}) = \{u \in L^2(\mathbb{T}) : \forall k<0,\ \hat{u}(k)=0\}.$$

<u>Global wellposedness:</u> $\forall s > \frac{1}{2}$, $\exists!\, u \in C(\mathbb{R}, H^s_+)$, $H^s_+ = H^s \cap L^2_+$,

$\forall u_0 \in H^s_+(S)$, $u(0) = u_0$,

$\cancel{\|u\|_{L^2}}$ $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, $\underbrace{\|u(t)\|_{L^4} = \|u_0\|_{L^4}}_{\text{Hamiltonian}}$.

$$(\underbrace{Du(t), u(t)}_{\sum_{k\geq 0} k|\hat{u}(k)|^2})_{L^2} = (Du_0, u_0)_{L^2}.$$

<u>Th^{m}</u> (S. Grellier, PG, 2010–2015).

① $\forall s > 1$, $u_0 \in H^s_+$, $\|u(t)\|_{H^s} \leq B(u,s)\, e^{B(u_0,s)|t|}$.

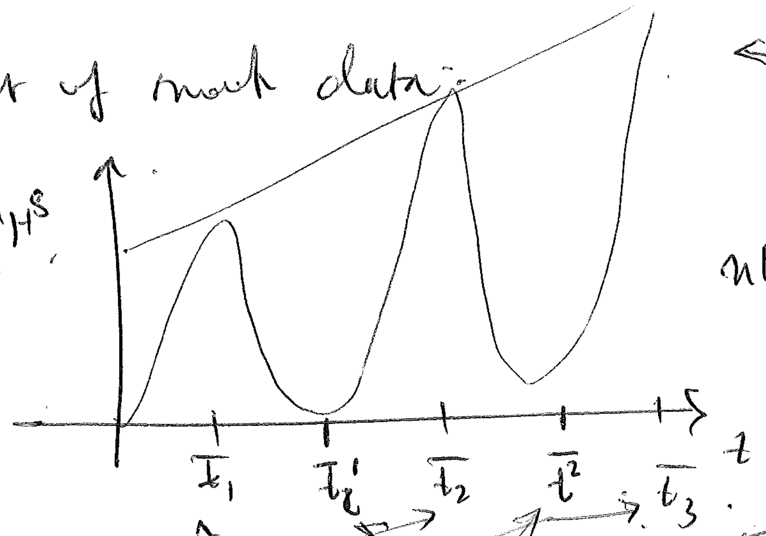
② Let $C^\infty_+ = \bigcap_s H^s_+ = C^\infty(\mathbb{T}) \cap L^2_+$. $\exists\, \mathcal{G} \subset C^\infty_+$ a dense G_δ subset

$\forall u_0 \in \mathcal{G}$, $\exists\ \bar{t}_n \to +\infty$, $t^n \to +\infty$. $\forall s > \frac{1}{2}$, $\forall p$,

$$\frac{\|u(\bar{t}_n)\|_{H^s}}{(\bar{t}_n)^p} \longrightarrow +\infty \quad \left(\begin{array}{l}\text{i.e., cannot find}\\ \text{pol. bound in gen.}\end{array}\right)$$

$u(t^n) \to u_0$ as $n \to \infty$ in C^∞_+.

For most of smooth data:



← goes to $+\infty$. Super polynomially!

$$u(t,x) = \sum_{k\geq 0} e^{ikx}\, \hat{u}(t,k)$$

$$\sum (k+1)|\hat{u}(t,k)|^2 \approx \text{cst}.$$

"wave turbulence".
forward–inverse
cascade

Transition to: high freq. low freq.

Related results:

1) O. Pocovnicu: Szegő, $\exists$ smooth sol^{n}s. (on the line)

Szegő on the line, $\forall s > \frac{1}{2}$; $\|u(t)\|_{H^s} \approx t^{2s-1}$, $t \to \infty$.

(This does not have the high/low freq switch as before).

2). H. Xu; 2013 $\quad i\partial_t u = \Pi_+(|u|^2 u) + \alpha\left(\int_{\mathbb{T}} u\right)$

(on the circle) $\quad \forall \alpha > 0$ $\exists$ smooth sol^{n}.

$\forall s > \frac{1}{2}$ $\quad \|u(t)\|_{H^s} \simeq e^{\alpha(2s-1)|t|}$.

3). An elementary example: forget $\Pi = \Pi_+$.

$i\partial_t u = |u|^2 u, \quad u(0,x) = u_0(x)$.

$u(t,x) = u_0(x)\, e^{-it|u_0(x)|^2}$.

$s > 0$, $t \to \infty$. $\|u(t,\cdot)\|_{H^s} \approx t^s$; i.e., turbulent sol^{n}s even in this context of an ODE.