Long time Sobolev Estimates for Hamiltonian PDEs. 27/07/2015.

Lecture 1:

Lecture 1: From NLS to Szegö eq^n.
Lecture 2: Integrability of Szegö eq^n.
Lecture 3: Turbulent sol^{n}s.

I) Nonlinear Schrödinger Equation.

(M,g) cpct Riem mfld., $\dim M \in \{1,2,3\}$.

Δ Laplace Beltrami on C^∞ and $\underbrace{H^{s+2}(M) \longrightarrow H^s(M)}_{\text{Sobolev Spaces}}$.

$u = u(t,x)$, $t \in \mathbb{R}, x \in M$.

$$(H)\quad \begin{cases} i\,\partial_t u + \Delta u = \underbrace{|u|^2 u}_{\text{nonlinearity}} \\ u(0,x) = u_0(x). \end{cases}$$

~~cubic defocus~~

"cubic defocusing NLS".

Energy: $E(f) = \frac{1}{2}\int_M |\nabla f(x)|^2 \, d\mu_g(x) + \frac{1}{4}\int_M |f(x)|^4 \, dx.$

$\partial_t u = X_E(u)$: $\forall h \in C^\infty(M)$, $\underbrace{\operatorname{Im}(h \mid X_E(u)) = \frac{d}{ds}\Big|_{s=0} E(u+sh)}_{(*)}$

where $(h_1 | h_2) = \int_M h_1 \overline{h_2}\, d\mu_g$.

$(*)$ defines (H). $X_E(u)$ in sense distribution, $(*)$ is the Hamiltonian. Changing E, ~~you can~~ and computing $(*)$ you can get lots of equations (H) with different nonlinearities.

Consequence: $E(u(t,\cdot)) = \text{const}$ for u satisfies H

$\nearrow$ Conservation laws.

N. Burq, N. Tzvetkov, P. G. 2004:

$\forall u_0 \in H^s(M)$ $(s \geq 1)$ $\exists!$ $u \in C(\mathbb{R}, H^s(M))$ sol^n to (H).

$\|u(t)\|_{H^1(M)} = O(1)$ $t \in \mathbb{R}$ (conservation).

General questions: (Bourgain, 2000, $M = \mathbb{T}^d$).

How big can $\|u(t,\cdot)\|_{H^s}$, $s > 1$ be? (as $t \to \infty$).

Def^n. u is a turbulent sol^n if for $s > 1$,

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{H^s(M)} = +\infty.$$

($M = \mathbb{R}\times\mathbb{T}^2$, noncpct, Yes. I.e, sol^{n}s are turbulent.)
Hani-

The case $d = 1$. $M = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

$$i\partial_t u + \partial_x^2 u = |u|^2 u.$$

Rmk. $E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|u\|_{L^4}^2$. Because $+|u|^2 u$.
This is the defocusing because $+$ in $E \Rightarrow$ no competition b/w two terms. If $-|u|^2 u$, then $E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^4}^2$, and can focus.

Theorem (Zakharov–Shabat, 1972).

There is no turbulent sol^n to the above equation.
$\forall s$ integer ≥ 1, $\exists$ F_s pol. function $\varphi^s \approx (\mathbb{R}^2)^3 \to \mathbb{R}$.
at most quadratic in $u^{(s-1)}$; $\int_{\mathbb{T}} \left[|u^{(s)}(x)|^2 + F^s(u(x), \ldots, u^{(s-1)}(x))\right] dx$
is a $\underbrace{\text{C.L.}}_{\text{Conservation law}}$

Sketch of Pf Zakharov–Shabat. $u \in C^\infty(\mathbb{T})$.

$$L_u = \begin{pmatrix} -D & \bar{u} \\ u & D \end{pmatrix} \quad D = \frac{1}{i}\frac{d}{dx},$$

$L_u : H^{s+1}(\mathbb{T}, \mathbb{C}^2) \to H^s(\mathbb{T}, \mathbb{C}^2)$, s.a. unbounded on $L^2(\mathbb{T}, \mathbb{C})$.

Lax pair. If u solves $-i\partial_t u + \Delta u = 2|u|^2 u$, (factor 2 avoids $\sqrt{2}$ later)

$$\partial_t L_{u(t)} = [B_{u(t)}, L_{u(t)}],$$

$$B_u = \begin{pmatrix} 2i\partial_x^2 - i|u|^2 & u' + 2u\partial_x \\ \bar{u}' + 2\bar{u}\partial_x & -2i\partial_x^2 + i|u|^2 \end{pmatrix}, \quad B_u^* = -B_u.$$

Consequence: $\exists$ family of unitary operators $\mathcal{U}(t) : L^2(\mathbb{T}, \mathbb{C}^2) \to L^2(\mathbb{T}, \mathbb{C}^2)$

$$L_{u(t)} = \mathcal{U}(t) L_{u(0)} \mathcal{U}(t)^*.$$

(solve the linear ODE $\dot{\mathcal{U}}(t) = B_{u(t)}\mathcal{U}(t)$, $\mathcal{U}(0) = \mathrm{Id}$ by anti-symmetry of B_u.)

for $h > 0$, $(I + h^2 L_u^2)^{-1} : L^2(\mathbb{T}, \mathbb{C}^2) \longrightarrow L^2(\mathbb{T}, \mathbb{C}^2)$.

$$\searrow H^s(\mathbb{T}, \mathbb{C}^2) \nearrow \text{ trace class!}$$

$\mathrm{Tr}\,(I + h^2 L_u^2)^{-1}$ is a conservation law $\forall h > 0$.

Expand this in powers of $h > 0$, as $h \to 0^+$.

$$\mathrm{Tr}\,(I + h^2 L_u^2)^{-1} \sim \frac{1}{h} \sum_{j \geq 0} h^j \underbrace{T_j(u)}_{\text{conservation laws!}}.$$

$$I + h^2 L_u^2 = I + h^2 D^2 + h^2 M(u), \quad M(u) = \begin{pmatrix} |u|^2 & i\bar{u}' \\ -iu' & |u|^2 \end{pmatrix}.$$

$$= P(x, hD, h).$$

$$P(x, \xi, h) = I + \xi^2 + h^2 M(u).$$

Claim: $\exists\, A_j(x,\xi)_{j\geq 0}$, $(I+h^2L_u^2)\left(\sum_{j\geq 0} h^j A_j(x,hD)\right)\sim I$.

$$A(x,D)f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}A(x,\xi)\hat{f}(\xi)\,d\xi\,,$$

$$f\in \mathcal{S}(\mathbb{R})\supset \mathcal{D}'(\mathbb{T}).$$

includes periodic distributions.

Recipe for A_j's: $(1+\xi^2)A_0 = I$, $(1+\xi^2)A_1 = 2i\xi\,\partial_x A_0$,

$$(1+\xi^2)A_j = 2i\xi\,\partial_x A_{j-1} + (\partial_x^2 M(x))A_{j-2}(x,\xi)\quad j\geq 2.$$

$$\operatorname{Tr}(A(x,hD)) = \frac{1}{2\pi h}\iint_{\mathbb{T}\times\mathbb{R}} \operatorname{tr} A(x,\xi)\,dx\,d\xi + O(h^\infty).$$

give conservation laws.

$$A_0 = \frac{1}{1+\xi^2},\quad A_1 = 0,\quad A_2 = \frac{-M(x)}{(1+\xi^2)^2}.$$

$$\int_{\mathbb{T}} \operatorname{tr} M(x)\,dx = 2\int_{\mathbb{T}}|u(x)|^2\,dx.$$

$$\frac{1}{2\pi}\int A_4\,dx\,d\xi \sim c\int \operatorname{tr} M^2.$$

$$T_6(u) = c\int_{\mathbb{T}} |u''|^2 + (\partial_x|u|^2)^2 + 6|u|^2|u'|^2 + 2|u|^6)\,dx.$$

Can proceed via induction. So the norms:

$$\int_{\mathbb{T}} [|u^s(x)|^2 + F^s(u(x),\dots,u^{(s-1)}(x))]\,d\mu(x) \text{ are c.l.!}$$
$$\parallel$$
$$T_{2s+2}(u).$$

Recall: $H^s = \{u:\ \partial^\alpha u\in L^2,\ |\alpha|\leq s\}\hookrightarrow L^\infty$ $s>\frac{d}{2}$.

$\Rightarrow$ hold and cannot be turbulent.

II) Story on $\mathbb{T}$: Fractional NLS: $i\partial_t u - |D|^\alpha u = |u|^2 u$.

$1 \le \alpha \le 2$, $u(0,x) = u_0(x)$

(a) Global well posedness and crude H^s.

Duhamel formulation: $u(t) = e^{-it|D|^\alpha} u_0 - i \int_0^t e^{-i(t-\tau)|D|^\alpha} [|u(\tau)|^2 u(\tau)] d\tau$

Fixed pt. argument: $H^s(\mathbb{T})$, $s > \frac{1}{2}$

Global existence: $E_\alpha(u) = \frac{1}{2} \int_{\mathbb{T}} |D|^\alpha u \, \bar{u} \, dx + \frac{1}{4} \int_{\mathbb{T}} |u|^4 dx$.

$\alpha > 1 \Rightarrow H^{\alpha/2} \subset L^\infty$. So in H^s, $s \geq \alpha/2$.

$\alpha = 1$, $H^{\frac{1}{2}} \not\subset L^\infty$. $E_1(u) \leftrightarrow \|u\|_{H^{\frac{1}{2}}}$.

<u>Lemma</u> (Brezis-Gallouët estimate).

$\forall s > \frac{1}{2}$. $\|u\|_{L^\infty(\mathbb{T})} \le C_s \|u\|_{H^{\frac{1}{2}}} \left[\log\left(2 + \frac{\|u\|_{H^s}}{\|u\|_{H^{\frac{1}{2}}}}\right) \right]^{\frac{1}{2}}$.

$s > \frac{1}{2}$, $\|u(t)\|_{H^s} \le \|u_0\|_{H^s} + \int_0^t \|u(s)\|^2_{L^\infty} \|u(\tau)\|_{H^s} d\tau$.

use Lemma so ~~need~~ quadratic term in the log term.

$\|fg\|_{H^s} \le \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}$.

$\|u(t)\|_{H^s} \le e^{e^{Bt}}$. (Os good).