M. Mitrea : Lecture 3. 28/07/2014.

On the way to mflds.

Meta-Thm: "All" previous results extend to variable coeff-kernels.

Eg, $b(x,z)$, $(x,z) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, smooth in x, odd in z & pos. hom. of deg $1-n$ (in z).

$$(Tf)(x) := \text{p.v.} \int_{\partial\Omega} b(x, x-y) f(y)\, d\sigma(y), \quad f \in L^p(\partial\Omega),$$
$1 < p < \infty$.

$\Omega \subset \mathbb{R}^n$ open, bdd. & $\partial\Omega$ ADR & UR.

Then, $T : L^p(\partial\Omega) \to L^p(\partial\Omega)$.

Compactness: $\Omega \subset \mathbb{R}^n$ open, bdd, $\partial\Omega$ ADR, Ω LJC.
$\nu \in VMO(\partial\Omega)$.

$$(Tf)(x) := \text{p.v.} \int_{\partial\Omega} b(x, x-y) \langle x-y, \nu(y) \rangle f(y)\, d\sigma(y).$$

b hom. ~~in x~~ of deg $-n$ here, but as before + even (b/c $x-y$ factor in $\langle\cdot,\cdot\rangle$ together makes it odd).

$\Rightarrow T \in Cpct(L^p(\partial\Omega))$

$P \in \underbrace{OPS^{-1}_{cl}}$. $\rightarrow$ $k(x,y)$ Schwartz kernel. of P.
~~principal~~ symbols for ΨDO. $\Rightarrow$ $(Tf)(x) := \text{p.v} \int_{\partial\Omega} k(x,y) f(y)\, d\sigma(y)$.
cl = classical. bdd in $p \in (1,\infty)$.

(M,g) C^2 cpt boundaryless Riem mfld, $g \in C^1$.

$\mathcal{E}, \mathcal{F} \to M$ Hermitian v. bundles $\in C^1$.

$L : \mathcal{E} \to \mathcal{F}$, 2nd order, strongly elliptic.

$L : W^{1,2}(M, \mathcal{E}) \to W^{-1,2}(M, \mathcal{F}) \Rightarrow$ Fredholm.

Rmk. We don't assume L is necessarily s.a.

Assume L is invertible $\Rightarrow$ This is not true for all operators, i.e. $L = \Delta_H$, Hodge-Laplace. In that case, Δ_H has unique continuation property, so consider $\Delta_H - V$ $\leftarrow$ potential

$V \neq 0$.

So, can "reduce" to invertible case.

$L^{-1} : W^{-1,2}(M) \to W^{1,2}(M)$.

$E(x,y) :=$ the Schwartz kernel of $L^{-1} \in \mathcal{E}$

Consider: $(Tf)(x) := \int_{\partial\Omega} \langle \nabla_y E(x,y), f(y) \rangle \, d\sigma(y)$.

$L(Tf) = 0$ in Ω, $\forall f : \partial\Omega \to \mathcal{E}$.

This will be "nice" (not necessarily C^∞ since metrics are C^1) away from diagonal.

$\Delta_H \rightsquigarrow E$. Sch. kernel of $\Delta_H - V$.

$g = g_{jk}\, dx^j \otimes dx^k$; then,

$$E(x,y) = e_0(x, x-y) + e_1(x, y)$$

where
$$e_0(x, z) = \frac{1}{\sqrt{g(x)}} \left(g^{jk}(x)\, (z_j z_k) \right)^{-(n-2)/2}.$$

$\forall\, \varepsilon > 0$, $|e_1(x,y)| \lesssim C_\varepsilon |x-y|^{-(n-3+\varepsilon)}$. (Pseudo + Microlocal).

<u>Moreover</u> $\| \mathcal{N}(\mathcal{T}f) \|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega)}$. $\forall f \in L^p(\partial\Omega)$.

$(\mathcal{T}f)\Big|_{\partial\Omega}^{n.t.}$ = messy. ("y" matters, ie in ∇_y in $(\mathcal{T}f)(x)$ formula
$(\mathcal{T}f)(x) = \int_{\partial\Omega} \langle \nabla_y E(x,y), f(y) \rangle\, d\sigma(y)$).

Sps $L = \tilde{D} D + Q$, D & $\tilde{D}$ 1st order, Q 0th order.

$\Delta_{LB} = -\operatorname{div} \operatorname{grad}$, $\Delta_H = -\nabla^* \nabla + \operatorname{Ric}$. via B–W.

Then, under this assumption:
$$\nabla_y = (-i)\, \operatorname{Sym}(D^T, \nu_y)\, \tilde{D}_y^T.$$

$$\Rightarrow (\mathcal{D}f)(x) := \int_{\partial\Omega} \langle (-i) \operatorname{Sym}(D^T, \nu_y)\, \tilde{D}_y^T E(x,y), f(y) \rangle\, d\sigma(y).$$
$x \in \Omega$.

$$(\mathcal{D}f)\Big|_{\partial\Omega}^{n.t.} = \left(\tfrac{1}{2} I + K\right) f, \quad \sigma\text{-a.e. on } \partial\Omega,$$
$f \in L^p(\partial\Omega)$, $1 < p < \infty$.

where, $(Kf)(x) := \text{p.v.}\int_{\partial\Omega} \langle (i)\,\mathrm{Sym}\,(D^T,\nu(y))\,\tilde{D}^T_y E(x,y), f(y)\rangle \, d\sigma(y)$

$x \in \partial\Omega$.

under $\Omega \subset M$, open, $\partial\Omega$ ADR & UR.

$\underline{\text{Thm}}$. L, Ω as above; $L = \tilde{D}D + Q$.

Associated with this quasi-factorisation, consider $\mathcal{D}$ and K as above. Then,

(I) $L(\mathcal{D}f) = 0$ in Ω, $\forall f \in L^p(\partial\Omega)$.

(II) $\|\mathcal{N}(\mathcal{D}f)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}$

(III) $K \in \mathcal{L}(L^p(\partial\Omega))$

(IV) $K \in \mathcal{L}(L^p_1(\partial\Omega))$.

(V) $(\mathcal{D}f)\Big|^{n.t}_{\partial\Omega} = (\tfrac{1}{2}I + K)f \quad \forall f \in L^p(\partial\Omega)$.

(VI) $\|\mathcal{N}(\nabla\mathcal{D}f)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p_1(\partial\Omega)}$.

(VII) $\int_\Omega |\nabla(\mathcal{D}f)(x)|^2 \,\mathrm{dist}(x,\partial\Omega)\, d\mathrm{vol}(x) \leq \int_{\partial\Omega} |f|^2 \, d\sigma$. Sq. funct est.

(VIII). $2 \leq p < \infty$. (p-version)

$$\int_{\partial\Omega} \sup_{\substack{x\in\partial\Omega \\ r>0}} \left(\frac{1}{r^{n-1}} \int_{B\cap\Omega} |\nabla \quad \right) \Big)$$

(IX) $|\nabla(\mathcal{D}f)|^2 \,\mathrm{dist}\,(x,\partial\Omega)\, dV(x)$ Carleson meas. if $f \in BMO(\partial\Omega)$.

(X). $\partial_{\tau_{xy}}(kf) = k(\partial_{\tau_{xy}} f) + [T, M_\nu](\partial_{\tau_{xy}} f)$.

$f \in L^p_1(\partial\Omega)$.

(XI) If k is compact in $L^p(\partial\Omega)$, and $\nu \in VMO(\partial\Omega)$.

$\Rightarrow$ k is cpct on $L^p_1(\partial\Omega)$.

The p.v. double layer associated with the writing of $\Delta_H = -\nabla^*\nabla + Ric$ does have the algebraic structure which guarantees compactness in $L^p(\partial\Omega)$

Rmk. Take background metric & perturbe

$$(Sf)(x) := \int_{\partial\Omega} \langle E(x,y), f(y) \rangle \, d\sigma(y). \quad x \in \Omega.$$

* single layer potential.

$$(Sf)\Big|^{n.t.}_{\partial\Omega} = (\mathcal{S}f)(x) \quad x \in \partial\Omega.$$

where $\mathcal{S}f(x) := \int_{\partial\Omega} \langle E(x,y), f(y) \rangle \, d\sigma(y)$. $\boxed{x \in \partial\Omega}$.

$\mathcal{S}: L^p(\partial\Omega) \to L^p_1(\partial\Omega)$,

but can't extract compact part + regular part $\left(\frac{1}{2}I + k\right)$ as in the $\mathcal{D}_A$ case.

Ω Lip domain in $\mathbb{R}^n$, $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

$$\int_{\partial\Omega} k(x-y) f(y) \, d\sigma(y) = \int_{\mathbb{R}^{n-1}} k\Big((x', \varphi(x')) - (y', \varphi(y'))\Big) \tilde{f}(y') \sqrt{1 + |\nabla\varphi(y')|^2} \, dy'.$$

$k(x'-y', \underbrace{\varphi(x')-\varphi(y')}_{})$

$$(D\varphi)(x')(x'-y') + \text{err}.$$

Assume further $\nabla\varphi \in L^\infty \cap VMO$, then,

$$(T^\# f)(x') := p.v.\int_{\mathbb{R}^{n-1}} k(x'-y', (D\varphi)(x'-y')).f(y')\,dy'!$$

$x \in \mathbb{R}^{n-1}$.

Then, $T^\# \in OP(L^\infty \cap VMO)S^0_{cl}$!

$OPX S^m_{cl}$ $\quad \|\partial^\alpha_x \partial^\beta_\xi a(x,\xi)\|_X \lesssim \langle\xi\rangle^{m-|\beta|}$

Moreover, Term (from err term), : $L^p(B) \to L^p(B)$ is compact $\forall B \subset \mathbb{R}^{n-1}$!

So, we can discuss problem modulo compact operators and $OP(L^\infty \cap VMO)S^0_{cl}$ is an algebra.

$\sigma(T) \Rightarrow$ symbol of $T^\#$, then,

Thm. If $\sigma(T)$ elliptic, then $T : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is Fredholm.

(happens if Ω bdd Lip-dom with $\nu \in VMO(\partial\Omega)$).

Single layer potential fits this picture? (after taking derivative of single layer pot).