M. Mitrea - Lecture 2. 30/07/2015.

Previous th^m [MMV] for. $\alpha = 0$:

$\Omega \subset \mathbb{R}^n$ open, bdd, Ω ADR. TFAE:

(I) $R_j 1 \in VMO(\partial\Omega)$ - vanishing mean osc.

(II) $\nu \in VMO(\partial\Omega)$. and $\partial\Omega$ is UR.

De Georgi - Federer outward unit normal (very weak).

(I) Analysis $\iff$ (II) GMT.

Rmk. $\partial\Omega$ UR $\Rightarrow$ $C: BMO(\partial\Omega) \to BMO(\partial\Omega)$.

Under $\partial\Omega$ ADR $\Rightarrow$ $C: C^\alpha(\partial\Omega) \to C^\alpha(\partial\Omega)$.

Since VMO can be realized as $\overline{\bigcup_\alpha \{ h \in C^\alpha \}}^{\|\cdot\|_{BMO}}$,

these two things imply $C: VMO(\partial\Omega) \to VMO(\partial\Omega)$.

Perturbation is true as well in the following sense

$$\sum_{j=1}^n \text{dist}(R_j 1, VMO(\partial\Omega)) \text{ "}\approx\text{" } \text{dist}_{BMO}(\nu, VMO(\partial\Omega)).$$

distance in $BMO(\partial\Omega)$.

Context; V. Maz'ya, M., T. Shaposhnikova.

$$\mathcal{L}u = \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta}(x) \partial^\beta u).$$

$\Omega \subset \mathbb{R}^n$. Lip. domain, $p \in (1,\infty)$, $s \in (0,1)$.

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega. \\ Tr(\partial^\gamma u) = f_\gamma \text{ in } \partial\Omega. \\ \{f_\gamma\}_{|\gamma| \le m-1} \in \dot{B}^{p,p}_{m-1+s}(\partial\Omega). \\ u \in B^{p,p}_{m-1+s+\frac{1}{p}}(\Omega). \end{cases}$$

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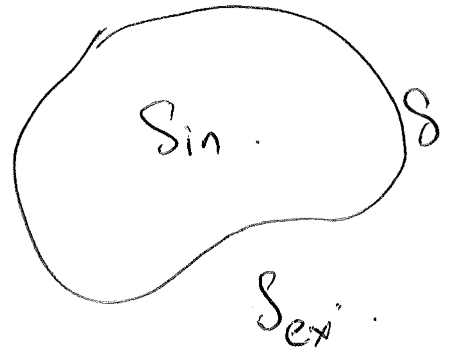
Question: Is this problem well-posed?

Answer: If $d(A_{\alpha\beta}; VMO(\Omega)) + \mathrm{dist}(\nu, VMO(\partial\Omega))$ is small! ~~the~~

Moreover: this is sharp – there exist counterexamples.

S_{in} & S_{ex}.

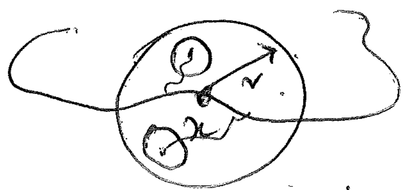


S_{in} – interior
S_{ex} – exterior

How do you discern the diff?

Upper half plane = you take for **granted** that you have a transversal direction. But when does this happen in general?

Def. Ω has a two sided **local jump condition** if



$\forall x \in \partial\Omega$, $\forall r > 0$, $\forall B$.

~~$\exists$ $B(cr)$ ... inside~~ B.

$\exists$ $B(cr, x_1)$ inside, $B(cr, x_2)$ outside s.t.

$\exists$ paths from centers x_1, x_2 to $\partial\Omega \cap B(r, x)$ in a "non-tangential" way (ie, dist from $\partial\Omega$ is uniform.

NTA $\Rightarrow$ 2 sided L.J.C.

Thm. $\Omega \subset \mathbb{R}^n$, open, bdd, $\partial\Omega$ ADR & L.J.C.

Then

$$\nu \in VMO(\partial\Omega) \iff \begin{cases} \left(I + \sum_{j=1}^{n} R_j^2\right) \text{ compact on } L^p \\ [R_j; R_j] \text{ compact on } L^p \end{cases}$$

Think of $\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n$. In this situation.

$I + \sum_{j=1}^n R_j^2 = 0$ and $[R_j, R_k] = 0$. In this case $\nu = \uparrow \nu$ _ _ _ and so, trivially $\nu \in VMO$.

Conversely, if the ⓪ condition holds, then $\partial\Omega = S^{n-1}$ or $\partial\Omega$ = plane $(n-1)$-dim. A rigidity theorem is also true → if the norms are small, then $\partial\Omega$ is not far from being S^{n-1} or $(n-1)$-plane.

$L^p_1(\partial\Omega)$ — How do you define?

Integration by parts formula on the boundary:

$\varphi, \psi \in C^1_c(\mathbb{R}^n)$, $j, k \in \{1, \dots, n\}$.

$$\int_{\partial\Omega} \varphi\,(\nu_j \partial_k - \nu_k \partial_j)\,\psi \, d\sigma = \int_{\partial\Omega} (\nu_k \partial_j - \nu_j \partial_k)\varphi \cdot \psi \, d\sigma.$$

True for ADR.

So, define $f \in L^p_1(\partial\Omega)$,

$$\int_{\partial\Omega} f\,(\partial_{\tau_{jk}} \varphi)\,d\sigma = \int_{\partial\Omega} f_{jk}\, \varphi \, d\sigma \quad \forall \varphi \in C^1_c(\mathbb{R}^n)$$

$f_{jk} = \partial_{\tau_{jk}} f$ ← "tangential ~~der~~ derivatives"

$\partial\Omega$ UR $\Rightarrow$ $K : L^p_1(\partial\Omega) \to L^p_1(\partial\Omega)$. hold!

Also, $\partial_{\tau_{jk}}(Kf) = K(\partial_{\tau_{jk}} f) + [T_{jk}, M_\nu]\, \partial_{\tau_{jk}} f$.

CZ on $\partial\Omega$. ↗ ↖ pointwise mult via ν.

If we know K compact on L^p, ~~then~~ this and $[T_{jk}, M_\nu]$ compact on L^p, this formula gives that K compact on L^p;

<u>Idea:</u> Theorem in the spirit of David-Semmes:

~~CZ ops SIO's are~~
$\forall$ CZ ops are L^p ~~bdd~~ bdd $\iff$ set is UR.

Replaced <u>bdd</u> with cpt. But, need to only consider subset of CZ ops. $\to$ No reason to expect this to be true for every CZ op.

<u>Th^{m}</u>. $\Omega \subset \mathbb{R}^n$, open, bdd, $\partial\Omega$ ADR, 2 LJC, $p \in (1,\infty)$. Consider $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$, even, positive homogenous of deg $-n$. Fix $\varepsilon > 0$.

$$(Tf) := \text{p.v.} \int_{\partial\Omega} \langle x-y, \nu(y) \rangle k(x-y) f(y)\, d\sigma(y).$$

$x \in \partial\Omega$.

for $f \in L^p(\partial\Omega)$.

Then, $\exists\, \delta(\varepsilon, p, \Omega, k) > 0$ s.t.
if $\mathrm{dist}_{BMO(\partial\Omega)}(\nu, VMO(\partial\Omega)) < \delta$

$\Rightarrow \mathrm{dist}(T, \mathrm{Comp}(L^p(\partial\Omega))) < \varepsilon$.

Also, consider:

$\tilde{k} \in C^\infty(\mathbb{R}^n \setminus \{0\})$, odd, pos hom deg $1-n$,

$(\tilde{T}f)(x) := \text{p.v.} \int_{\partial\Omega} \tilde{k}(x-y) f(y)\, d\sigma(y)$ $x \in \partial\Omega$.

for $f \in L^p(\partial\Omega)$.

Then, $\operatorname{dist}(\nu, VMO(\partial\Omega)) < \delta \Rightarrow$

$\operatorname{dist}([\tilde{T}, M_{\nu}], \operatorname{Comp}(L^p(\partial\Omega))) < \varepsilon$.

Conversely, if $T, \tilde{T}$ cpt on L^p, for, say

$k(z) = \frac{1}{|z|^n}$ and $\tilde{k}(z) = \frac{z_i}{|z|^n}$. $1 \leq i \leq n$.

Then, necessarily, $\nu \in VMO(\partial\Omega)$.

How to put a weight?

Thm (D_p). $\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u\big|_{\partial\Omega}^{n.t.} = f \in L^p(\partial\Omega) \\ \mathcal{N}u \in L^p(\partial\Omega) \end{cases}$.

is well-posed if $\Omega \subset \mathbb{R}^n$ open, bdd, $\partial\Omega$ ADR, Ω LJC & $\nu \in VMO(\partial\Omega)$.

Possible to replace $L^p_1(\partial\Omega)$ with $L^p_w(\partial\Omega)$.

$w \in A_p(\partial\Omega)$: $\sup_{Q \subset \partial\Omega} \left(⨍_Q w\, d\sigma\right)\left(⨍_Q w^{-\frac{1}{p-1}}\, d\sigma\right)^{p-1} < \infty$

All CZ ops are L^p_w bdd for weighted L^p. Question is for compactness: Yes — this is true.

$1 < p < \infty$, $w \in A_p(\partial\Omega)$.

$\Rightarrow \exists\, \theta \in (0,1)$ s.t.

$w^{1/\theta} \in A_p(\partial\Omega)$.

$$\Big[\underset{\substack{\uparrow \\ \text{know } K \text{ cpct.}}}{L^p(\partial\Omega)},\ \underset{\substack{\uparrow \\ K \text{ bdd.}}}{L^p_{w^{1/\theta}}(\partial\Omega)} \Big]_{\theta, p} = L^p_w(\partial\Omega).$$

Real method

Famous interpolation theorem; Michael (Sweden)?
gives interpolation from to there K compact!
In particular $L^p_w(\partial\Omega)$