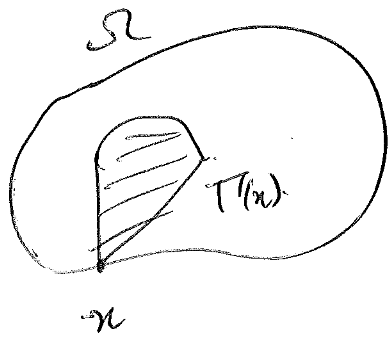
SIO + ΨDO. M. Mitrea. 24/07/2015.

Fabes - Jodeit - Rivière.

(P) $\left[\ \Delta u = 0, \quad u\big|_{\partial\Omega} = f \in L^p(\partial\Omega),\right.$

Even if $u \in C^\infty(\Omega)$, what does $u\big|_{\partial\Omega}$ mean? it's defined on Ω, not $\overline{\Omega}$.

$u\big|_{\partial\Omega}$ means <u>non-tangentially</u>,



$$\Gamma(x) := \{ y \in \Omega : |x-y| < 2\,\mathrm{dist}(y, \partial\Omega) \}.$$

$$u\big|^{n.t}_{\partial\Omega}(x) := \lim_{\Gamma(x) \ni y \to x} u(y).$$

So, full formulation of (P), need to add.

$$\mathcal{N}u \in L^p(\partial\Omega).$$

where $\mathcal{N}u(x) = \|u\|_{L^\infty(\Gamma(x))}$.

Another problem: $\Delta u = 0$ in Ω, $u\big|^{n.t}_{\partial\Omega} = f \in L^p_1(\partial\Omega)$ has some tangential derivatives.

So ask not only $\mathcal{N}u \in L^p(\partial\Omega)$

but also $\mathcal{N}(\nabla u) \in L^p(\partial\Omega)$.

These are well-posed $\forall p \in (1,\infty)$. if $\Omega \subset \mathbb{R}^n$. bdd C^1-domain.

Closely followed by Dahlberg:

$$(\mathcal{D}g)(x) := \frac{-1}{w_n} \int_{\partial\Omega} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} g(y)\, d\sigma(y). \quad x \in \Omega.$$

$g : \partial\Omega \to \mathbb{R}$.

Candidate: $u = \mathcal{D}g$, where g TBA.

$\| \mathcal{N}(\mathcal{D}g) \|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}$.

$(\mathcal{D}g)\Big|_{\partial\Omega}^{n.t.} = \left(\frac{1}{2} I + k\right) g$ σ-a.e. on $\partial\Omega$.

$$(kg)(x) := \text{p.v.} \frac{-1}{w_{n-1}} \int_{\partial\Omega} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} g(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

So, Solvability: $u := \mathcal{D}\left(\left(\frac{1}{2}I + k\right)^{-1} f\right)$

given initial datum $f \in L^p(\partial\Omega)$.

So, Q: $\left(\frac{1}{2}I + k\right)^{-1} \in \mathcal{L}(L^p(\partial\Omega))$?

$k = \sum_{j=1}^n R_j M \nu_j$.

$$(R_j g)(x) = \text{p.v.} \left(\frac{-1}{w_{n-1}}\right) \int_{\partial\Omega} \frac{x_j - y_j}{(x-y)^n} g(y)\, d\sigma(y).$$

Reisz-transform.

Q. R_j bdd on $L^p(\partial\Omega)$?

$k_j(z) = \frac{z_j}{|z|^n}$. associated kernels.

$k \in C^\infty(\mathbb{R}^n \setminus \{0\})$, odd, ← Homogeneity on boundary!

$|(\partial^\alpha k)(z)| \lesssim |z|^{1-n-|\alpha|}$ $\forall \alpha \in \mathbb{N}_0^k$.

G. David / S. Semmes 1991.

All above SIO's hold on $L^2(\partial\Omega)$.

$\Longleftrightarrow$ $\partial\Omega$ is Uniformly Rectifiable set. (UR).

under background assumption that $\partial\Omega$ is ADR (Ahlfors-David reg.).

ADR means: $\Omega \subset \mathbb{R}^n$ open, $\partial\Omega$ ADR

means $\forall x \in \partial\Omega$, $\forall r \in (0, \operatorname{diam} \Omega)$.

$$\mathcal{H}^{n-1}(\partial\Omega \cap B(x,r)) \approx r^{n-1}.$$

<u>I.e.</u> $(\partial\Omega, |\cdot - \cdot|, \sigma := \mathcal{H}^{n-1} \llcorner \partial\Omega)$

space of homogeneous type. (SHT).

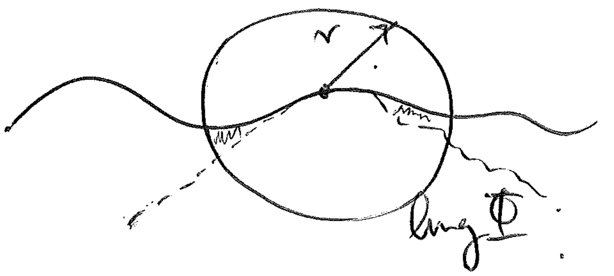
<u>Def^n</u>. $\partial\Omega$ is UR $\overset{def}{\Longleftrightarrow}$ $\exists\, \varepsilon, M > 0$,

s.t. $\forall x \in \partial\Omega$ $\forall r \in (0, \operatorname{diam} \partial\Omega)$.

$\exists\, \Phi : B_r^{n-1} \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ Lipschitz with $\operatorname{Lip}(\Phi) \le M$.

~~with~~ such that:

$$\mathcal{H}^{n-1}(\partial\Omega \cap B(x,r) \cap \Phi(B_r^{n-1})) \ge \varepsilon r^{n-1}.$$



Bddness of R_j $\subset L^2$, $j = 1, \dots, n$ $\Rightarrow$ UR.

<u>Th^m</u>: $\forall \varepsilon > 0$, $(R_{j,\varepsilon}\, g)(x) := \displaystyle\int_{\substack{|x-y|>\varepsilon \\ y \in \partial\Omega}} \frac{x_j - y_j}{|x-y|^n}\, g(y)\, d\sigma(y)$, $x \in \partial\Omega$

is bdd in $L^2(\partial\Omega)$, $y \in \partial\Omega$ uniformly in ε $\Rightarrow$ $\partial\Omega \in$ UR.

[S. Hofmann, M., M. Taylor]:

$\partial\Omega$ is UR $\Rightarrow$ $\lim_{\varepsilon\to 0}(T_\varepsilon g)(x)$ exist σ-a.e.

for any T_ε truncated SIO.

<u>Remark</u> Always background assumption that $\partial\Omega$ is ADR.

~~Moreover, ...~~ David-Semmes result does not say that R_j exists in p.v. sense, but just that $R_{j,\varepsilon}$ exist with bound indep of ε.

$\partial\Omega$ ADR, $\forall\alpha\in(0,1)$, $R_j: C^\alpha(\partial\Omega)\to(C^\alpha(\partial\Omega))^*$.

$$\langle R_j f, g\rangle = \frac{1}{2}\iint_{\partial\Omega}\frac{x_j-y_j}{|x-y|^n}\,[f(x)g(y)-f(y)g(x)]\,d\sigma(x)\,d\sigma(y).$$

NTV Thm: $\partial\Omega$ UR $\Leftrightarrow$ $R_j 1\in BMO(\partial\Omega)$.

<u>Q</u>. $\alpha\in(0,1)$ and $\exists$ps. $R_j 1\in C^\alpha(\partial\Omega)$

$\overset{(?)}{\Longrightarrow}$ Ω is a $C^{1,\alpha}$-domain.

<u>Yes!</u> joint w/ Dorina Mitrea & J. Verdera.

<u>Thm</u>. (MMV). $\partial\Omega$ ADR. (and $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial_*\Omega)=0$.) technical ass.
fix $\alpha\in(0,1)$.

TFAE:

(I) $R_j 1\in C^\alpha(\partial\Omega)$.

(II) $R_j: C^\alpha(\partial\Omega)\to C^\alpha(\partial\Omega)$ hold.

(III) $\forall$ P polynomial, odd, hom. in $\mathbb{R}^n$.

$$(T_P f)(x) := \text{p.v.}\int_{\partial\Omega}\frac{P(x-y)}{|x-y|^{n-1+\deg(P)}}\,f(y)\,d\sigma(y).\quad x\in\partial\Omega.$$

(4) Ω is a $C^{1,\alpha}$ domain.

Pf. Look at Clifford alg. $e_1, \dots, e_n$, $e_j^2 = -1, \forall j$,

$e_j \odot e_k = -e_k \odot e_j$, $\forall j \neq k$.

$\mathbb{R}^n \ni x = (x_1, \dots, x_n) \equiv \sum x_j e_j \in$ ~~Cliff~~ $Cl(n)$

$$Cf(x) := \text{p.v.} \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y) \, d\sigma(y).$$

$$\Rightarrow, \quad \nu = -\frac{1}{4} C\left(\sum_{j=1}^n R_j(1) e_j\right).$$

Point: Indep of reg of $\partial\Omega$, $Cf: C^\alpha \to C^\alpha$,

So, if $R_j(1) \in C^\alpha$, then $\sum_{j=1}^n R_j(1) e_j \in C^\alpha$.

and $Cf: C^\alpha \to C^\alpha$ so $\nu \in C^\alpha$.

Ω L.F.P., then $\underset{\Omega}{\Omega \text{ Lip}} \Leftrightarrow \exists \vec{h} \in C^0(\partial\Omega)$

ess inf $\langle \vec{h}, \nu \rangle \geq k > 0$ on $\partial\Omega$

Ω C^1 domain $\Leftrightarrow \nu \in C^0(\partial\Omega)$,

in fact $\vec{h} = \nu$ in

Ω $C^{1,\alpha}$ dom $\Leftrightarrow \nu \in C^\alpha(\partial\Omega)$.