Giuseppe Mingione. - Nonlinear CZ theory. 08/06/2015.

- Classical picture: $\Delta u = f$ in $\mathbb{R}^n$

  Then $f \in L^q \Rightarrow \nabla^2 u \in L^q$ $1 < q < \infty$ (failure at $q = 1, \infty$).

  Really what happens: $\nabla^2 u \simeq \Delta u$.

  Sobolev embedding: $Du \in L^{\frac{nq}{n-q}}$. $q < n$.

- SIO approach. $u(x) \approx \int G(x,y) f(y)$. G - Green's function.

  Differentiation yields: $D^2 u(x) = \int K(x,y) f(y)\, dy$.

  K singular integral kernel.

- Cancellation is the thing that saves you.

- Initial conditions $\|\hat{k}\|_{L^\infty} \le B$. $\wedge$: FT

- Hörmander $\int_{|x| \ge 2|y|} |k(x-y) - k(x)|\, dx \le B$ $\forall y \in \mathbb{R}^n$.

- Instead of diff. u twice, just once gives

  $$I_\beta(g)(x) = \int \frac{g(y)}{|x-y|^{n-\beta}}\, dy \quad \beta \in [0,n)$$ (Fractional integrals).

  and $|Du(x)| \lesssim I_1(|f|)(x)$.

  Also $I_\beta : L^q \longrightarrow L^{\frac{nq}{n-\beta q}}$ $\beta q < n$. (original Sobolev funcs).

- Fractional integrals <u>do not</u> require cancellation, only size.

Formally "simplify" divergence

$$\Delta u = \text{"}\cancel{\text{div}}\,\nabla u = \cancel{\text{div}}\, F\text{"}$$

$$\Rightarrow \quad \nabla u = F.$$

First year Engineers approach.

$$F \in L^q \Rightarrow Du \in L^q \qquad q > 1.$$

~~Def~~

• Alt approach:

Define $T : F \longmapsto T(F)$:= gradient of solⁿ to $\Delta u = \operatorname{div} F$.

$\Rightarrow \quad T : L^2 \to L^2$ and $T : L^\infty \to BMO$. (via regularity)
hard part.

Campanato–Stampacchia Interpolation.

$\Rightarrow \quad T : L^q \to L^q$.

• Linear to nonlinear: $\quad \operatorname{div} a(Du) = 0$.

Iwaniec '83: $\operatorname{div}(|Du|^{p-2} Du) = \operatorname{div}(|F|^{p-2} F)$

$F \in L^q \Rightarrow Du \in L^q \quad p \leq q < \infty$.

DiBenedetto & Manfredi '93: $F \in BMO \Rightarrow Du \in BMO$.

• Sharp maximal operators are at the heart ~~Paradise~~ of all of this.

Parabolic Case

Acerbi & Min. 07:

$$u_t - \operatorname{div}\left(|Du|^{p-2} Du\right) = \operatorname{div}\left(|F|^{p-2} F\right) \quad \text{in } \Omega \times (0,T).$$

$p > \frac{2n}{n+2}$.

$\Rightarrow$. $F \in L^q_{loc} \Rightarrow Du \in L^q_{loc}$. $p \leq q < \infty$.

Harm. Anal. free approach to non lin. CZ estimates.

$$\left(\int_{Q_R} |Du|^p \, dz \right)^{\frac{1}{p}} \leq c \left[\left(\fint_{Q_{2R}} |Du|^p \right)^{\frac{1}{p}} + \left(\fint_{Q_{2R}} |F|^q \right)^{\frac{1}{p}} + 1 \right]^{p/2}$$

From time. part.

not present in Elliptic. This is due to lack of scaling in parabolic case.

Intrinsic geometry of DiBenedetto:

$$Q^\lambda_\rho(z_0) = B(x_0, \rho) \times \left(t_0 - \lambda^{2-p}\rho^2, t_0\right).$$

adds additional diffusion.

$|Du| \approx \lambda$. on $Q^\lambda_\rho(z_0)$.