Shanmugalingam 08/06/2014.

Classify metric spaces by ℓ^p spaces.

Cantor type sets: cpct, tot. disconnected, uniformly perf.

By bi Hölder change in metric $\Longrightarrow$ X ultrametric space.

$$d(x,z) \le \max\{d(x,y), d(y,z)\}.$$

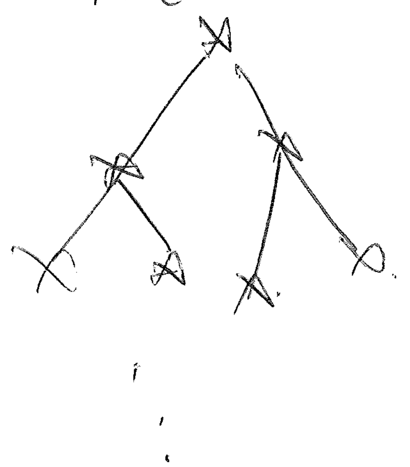
($\Rightarrow$ disconnected).

No rectifiable curves: so Sob. spaces are L^p, so trivial.

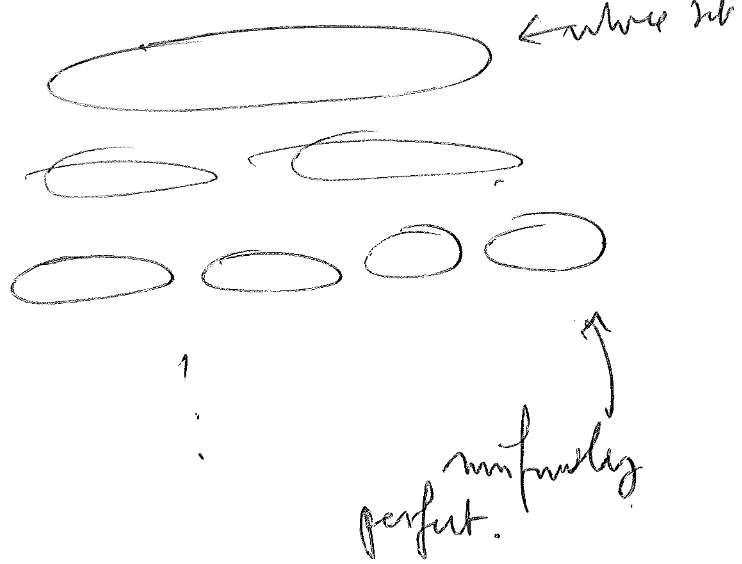
But Besov spaces are ok if using Ahlfors-reg.

Q. Are these spaces preserved under quasimetric maps?

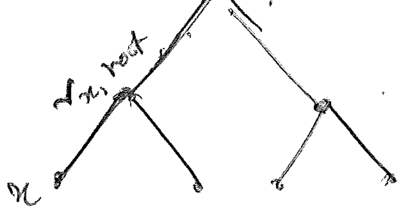
Tree



Set.



End up with infinite tree, uniformizing tree:
by drawing each generation with smaller lengths.

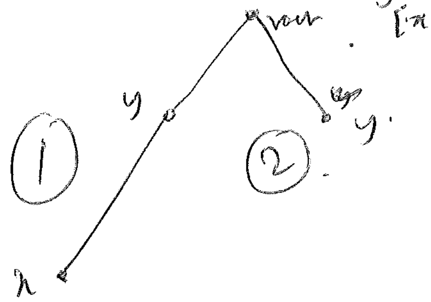


$$d_\varepsilon(x, root) = \int_{\gamma_{x,root}} e^{-\varepsilon|w|}\, d|w|.$$

$|x|$ = min # of vertices to get to root.

Given $x,y \in X$, $[x,y]$! geodesic from x to y.

$$d_\varepsilon(x,y) = \int_{[x,y]} e^{-\varepsilon t}\,dt.$$



① $d_\varepsilon(x,y) = \frac{1}{\varepsilon}\left[e^{-\varepsilon|y|} - e^{-\varepsilon|x|}\right]$. $[x,y]$

② $d_\varepsilon(x,y) = \frac{1}{\varepsilon}\left[(1-e^{-\varepsilon|y|}) + (1-e^{-\varepsilon|x|})\right]$

(X, d_ε) is a bounded metric space, $\operatorname{diam}_\varepsilon(X) = \frac{2}{\varepsilon}$ ← at least 2 leads at each vertex.

X has degree K uniformly.

Measures on (X, d_ε)

$\beta > \ln(K)$, $d\mu_\beta(x) = e^{-\beta|x|}\,d|x|$, $\mu_\beta(A) = \int_A e^{-\beta|x|}\,d|x|$

μ_β is doubling for bdd K.

$$\mu(B(x,r)) \simeq \begin{cases} r & \text{if } r \le \frac{e^{-\varepsilon|x|}}{\varepsilon} \\ r^{\beta/\varepsilon} & \text{if } r \ge \frac{e^{-\varepsilon|x|}}{\varepsilon} \end{cases}$$

$y \in B(x,R)$, $0 < r < R$,

$$\Rightarrow \frac{\mu(B(y,r))}{\mu(B(x,R))} \ge \frac{1}{C}\left(\frac{r}{R}\right)^{\max\{1, \beta/\varepsilon\}}.$$

$(X, d_\varepsilon, \mu_\beta)$ supports 1-Poincaré inequality.

$$\int_B |u - u_B|\,d\mu_\beta \le C\,\mathrm{rad}(B) \int_B g_u\,d\mu_\beta$$

↑ upper gradient.

$$g_u(x) := e^{\varepsilon|x|}|u'(x)|.$$

$(\partial X, d_\varepsilon)$ is Hölder equivalent to the original Cantor set.

$\xi \in \partial X \Rightarrow d_\varepsilon(\eta, \xi) = \frac{1}{\varepsilon} e^{-\varepsilon |x(\eta,\xi)|}$. $\boxed{Q = \frac{\ln(k)}{\varepsilon}}$

Hausdorff measure.

k^n gen.

k^n gen has k^n vertices.

$k^n \nu(B) \approx \text{radius}(B)^Q$.

$k^n e^{-\varepsilon n Q} \approx$.

and $(\partial X, d_\varepsilon)$ Ahlfors $\frac{\ln(k)}{\varepsilon}$-regular

Besov Spaces: $B^\theta_{p,p}(\partial X) = \{ f : \partial X \to \mathbb{R},\ \|f\|_{B^\theta_{p,p}(\partial X)} < \infty \}$.

$$\|f\|_{B^\theta_{p,p}(\partial X)} = \int_{\partial X}\int_{\partial X} \frac{|f(y) - f(x)|^p}{d(x,y)^{\theta p}\, \nu(B(x, d(y,x)))}\, d\mu(x)\, d\mu(y).$$

Why go through all of this?

Euclidean results: Ahlfors reg subset of $\mathbb{R}^n$,

Besov space is trace-class of Sobolev space.

Let $N^{1,p}(X)$ be Sobolev space on X, then.

Thm: $\exists$ bdd linear operator. $0 < \theta \leq 1 - \frac{\beta/\varepsilon - Q}{p}$.

$Tr : N^{1,p}(X) \longrightarrow B^\theta_{p,p}(\partial X)$.

$\theta \geq 1 - \frac{\beta/\varepsilon - Q}{p}$ $\quad Ex : B^\theta_{p,p}(\partial X) \longrightarrow N^{1,p}(X)$.

Rmk This is sharp!

$\underline{\mathrm{Th}^{m}}$: $\quad \varphi : \partial X \xrightarrow{\eta\text{-qs}} \partial Y$.

$Q_X \qquad Q_Y$ (~~dim~~).

$$\eta(t) = \begin{cases} At^{\alpha_1} & \text{if } t > 1 \\ At^{\alpha_2} & \text{if } t \leq 1 \end{cases} \qquad \alpha_2 \geq \alpha_1.$$

$x, y, z \in \partial X$.

$$\frac{d(\varphi(x), \varphi(y))}{d(\varphi(x), \varphi(z))} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right).$$

$$B_{p,p}^{Q_Y}(\partial Y) \xhookrightarrow{\varphi_\#} B_{p,p}^{Q_X}(\partial X) \xhookrightarrow{\varphi^{-1}_\#} B_{p,p}^{Q_Y}(\partial Y).$$